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ON THE RATES OF CONVERGENCE OF SIMULATION-BASED OPTIMIZATION ALGORITHMS FOR OPTIMAL STOPPING PROBLEMS

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In this paper, we study simulation-based optimization algorithms for solving discrete time optimal stopping problems. Using large deviation theory for the increments of empirical processes, we derive optimal convergence rates for the value function estimate and show that they cannot be improved in general. The rates derived provide a guide to the choice of the number of simulated paths needed in optimization step, which is crucial for the good performance of any simulation-based optimization algorithm. Finally, we present a numerical example of solving optimal stopping problem arising in finance that illustrates our theoretical findings.

1. Introduction. Let us consider a discrete time optimal stopping problem of the form:

$$V^* = \sup_{1\le\tau\le K} \mathrm{E}[Z_\tau], \tag{1.1}$$

where τ is a stopping time taking values in the set $\{1,\dots,K\}$ and $(Z_k)_{k\ge 0}$ is a Markov chain. In most cases, the expectation in (1.1) cannot be computed in a closed form and we have to approximate it numerically in order to find V^*. In this paper, we study a simulation-based approach to the optimal stopping problem (1.1). The basic idea is simple—for any τ from a feasible subset of the set of all stopping times valued in $\{1,\dots,K\}$, a random sample from Z_τ of the size M is generated and the expected value function is approximated by the corresponding sample average function. The resulting sample average optimization problem is then solved and a suboptimal policy τ_M is obtained. By sampling from Z_{τ_M} and averaging once again, we get a low biased approximation for V^* denoted by $V_{M,N}$, where N is the size of the second sample. The idea of using sample average approximations for solving the optimal stopping problem (1.1) is a natural one and was successfully used by practioneers over the years. Such an approach is, for example, popular in the context of a Bermudan option pricing problem in finance [see, e.g., Glasserman (2003), Section 8.2]. The main issues we are going

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to study in this work are how fast $V_{M,N}$ converge to V^* as $M, N \to \infty$ and what the optimal relation between M and N is that minimizes the computational costs. To the best of our knowledge, these problems are new and have not been studied before.

To get more insight on what kind of convergence rates one can expect, let us start with the general stochastic programming problem:

$$h^* := \min_{\theta \in \Theta} \mathrm{E}_{\mathrm{P}}[h(\theta, \xi)], \tag{1.2}$$

where Θ is a subset of $\mathbb{R}^m$, ξ is a $\mathbb{R}^d$ valued random variable on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and $h : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$. Draw an i.i.d. sample $\xi^{(1)}, \dots, \xi^{(M)}$ from the distribution of ξ and define

$$h_M := \min_{\theta \in \Theta} \left[\frac{1}{M} \sum_{m=1}^{M} h(\theta, \xi^{(m)}) \right].$$

It is well known [see, e.g., Shapiro (1993)] that under very mild conditions it holds $h_M - h^* = O_{\mathrm{P}}(M^{-1/2})$. In their pioneering work, Shapiro and Homem-de-Mello (2000) showed that in the case of discrete random variable ξ and a convex function h, the convergence of h_M to h^* can be much faster than $M^{-1/2}$, making simulation-based approach particularly efficient in this situation. Turn now back to the problem (1.1). Since the random variable τ takes only discrete values, one can ask whether the simulation-based methods in the case of discrete time optimal stopping problem (1.1) can be as efficient as in the case of (1.2) with discrete r.v. ξ. In this work, we give an affirmative answer to this question by deriving the optimal rates of convergence for the conditional mean of $V_{M,N}$ given a sample of size M, and showing that these rates are, under some mild conditions, faster than $M^{-1/2}$. This fact has an important practical implication since it indicates that M, the number of simulated paths used in the optimization step, can be taken much smaller than N, the number of paths used to compute the final estimate $V_{M,N}$, leading to a significant reduction of computational costs in the optimization step.

The paper is organized as follows. In Section 2, some notation are introduced and the optimal stopping problem is rigorously stated. In Section 3, main results are formulated and discussed. Some applications are presented in Section 4. Proofs of the main results are collected in Section 5. Section 6 contains the proofs of some lemmas needed for the proof of the main results. Finally, in Section 7 several exponential inequalities for the increments of empirical processes are presented.

2. Main setup.

Let us consider a Markov chain $X = (X_k)_{k \geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k \geq 0}, \mathrm{P}_x)$ and taking values in a measurable space $(E, \mathcal{B})$, where for simplicity we assume that $E = \mathbb{R}^d$ for some $d \geq 1$ and $\mathcal{B} = \mathcal{B}(\mathbb{R}^d)$ is the Borel σ-algebra on $\mathbb{R}^d$. It is assumed that the chain X starts at x under P_x for some $x \in E$. We also assume that the mapping $x \mapsto P_x(A)$

is measurable for each $A \in \mathcal{F}$. Fix some natural number $K > 0$. Given a set of measurable functions $G_k : E \mapsto \mathbb{R}$, $k = 1, \ldots, K$, satisfying

$$\mathrm{E}_x\Big[\sup_{1 \le k \le K} |G_k(X_k)|\Big] < \infty$$

for all $x \in E$, consider the optimal stopping problems:

$$V_k^*(x) := \sup_{k \le \tau \le K} \mathrm{E}_{k,x}[G_\tau(X_\tau)], \qquad k = 1, \ldots, K, \tag{2.1}$$

where for any $x \in E$, the expectation in (2.1) is taken w.r.t. the measure $\mathrm{P}_{k,x}$ such that $X_k = x$ under $\mathrm{P}_{k,x}$ and the supremum is taken over all stopping times τ with respect to $(\mathcal{F}_n)_{n \ge 0}$. Introduce the stopping region $\boldsymbol{\mathcal{S}}^* = \mathcal{S}_1^* \times \cdots \times \mathcal{S}_K^*$ with $\mathcal{S}_K^* = E$ by definition and

$$\mathcal{S}_k^* := \{x \in E : V_k^*(x) \le G_k(x)\}, \qquad k = 1, \ldots, K - 1.$$

Introduce also the first entry times τ_k^* into $\boldsymbol{\mathcal{S}}^*$ by setting

$$\tau_k^* := \tau_k(\boldsymbol{\mathcal{S}}^*) := \min\{k \le l \le K : X_l \in \mathcal{S}_l\}.$$

It is well known [see, e.g., Peskir and Shiryaev (2006)] that the value functions $V_k^*(x)$ satisfy the so called Wald–Bellman equations

$$V_k^*(x) = \max\{G_k(x), \mathrm{E}_{k,x}[V_{k+1}^*(X_{k+1})]\}, \qquad k = 1, \ldots, K - 1, \tag{2.2}$$

with $V_K^*(x) \equiv G_K(x)$ by definition. The Wald–Bellman equations (2.2) imply that the sets $\mathcal{S}_k^*$ can be also defined as

$$\mathcal{S}_k^* = \{x \in E : \mathrm{E}_{k,x}[V_{k+1}^*(X_{k+1})] \le G_k(x)\}, \qquad k = 1, \ldots, K - 1. \tag{2.3}$$

Moreover, the stopping times τ_k^* are optimal in (2.1), that is,

$$V_k^*(x) = \mathrm{E}_{k,x}[G_{\tau_k^*}(X_{\tau_k^*})], \qquad k = 1, \ldots, K.$$

Let $(X_k^{(m)})_{k=0,\ldots,K}, m = 1, \ldots, M$, be M independent Markov chains with the same distribution as X all starting from the point $x \in E$. We can think of $(X_k^{(1)}, \ldots, X_k^{(M)})$, $k = 0, \ldots, K$, as a new process defined on the product probability space equipped with the product measure $\mathrm{P}_x^{\otimes M}$. Let $\mathfrak{B}$ be a collection of sets from the product σ-algebra

$$\mathcal{B}^K := \underbrace{\mathcal{B} \otimes \cdots \otimes \mathcal{B}}_{K}$$

that contains all sets $\boldsymbol{\mathcal{S}} \in \mathcal{B}^K$ of the form $\boldsymbol{\mathcal{S}} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_{K-1} \times E$ with $\mathcal{S}_k \in \mathcal{B}, k = 1, \ldots, K - 1$. Here, we take into account the fact that the stopping set $\mathcal{S}_K$ coincides with E. Let $\mathfrak{S}$ be a subset of $\mathfrak{B}$. Define

$$\boldsymbol{\mathcal{S}}_M := \arg\sup_{\boldsymbol{\mathcal{S}} \in \mathfrak{S}} \left\{ \frac{1}{M} \sum_{m=1}^{M} G_{\tau_1(\boldsymbol{\mathcal{S}})}\big(X_{\tau_1(\boldsymbol{\mathcal{S}})}^{(m)}\big) \right\}. \tag{2.4}$$

The stopping rule

$$\tau_M := \tau_1(\mathcal{S}_M) = \min\{1 \le k \le K : X_k \in \mathcal{S}_{M,k}\}$$

is generally suboptimal and therefore the corresponding Monte Carlo estimate

$$V_{M,N} := \frac{1}{N}\sum_{n=1}^{N} G_{\tau_M^{(n)}}\bigl(\widetilde{X}^{(n)}_{\tau_M^{(n)}}\bigr) \tag{2.5}$$

with

$$\tau_M^{(n)} := \min\{1 \le k \le K : \widetilde{X}_k^{(n)} \in \mathcal{S}_{M,k}\}, \qquad n = 1, \dots, N,$$

based on a new, independent of $(X^{(1)}, \dots, X^{(M)})$ set of trajectories

$$\bigl(\widetilde{X}_0^{(n)}, \dots, \widetilde{X}_K^{(n)}\bigr), \qquad n = 1, \dots, N,$$

is low biased, that is, it fulfills

$$V_M := \mathrm{E}_x\bigl[V_{M,N} | X^{(1)}, \dots, X^{(M)}\bigr] \le \sup_{\mathcal{S} \in \mathfrak{S}} \mathrm{E}_x\bigl[G_{\tau_1(\mathcal{S})}\bigl(X_{\tau_1(\mathcal{S})}\bigr)\bigr] \le V^* \tag{2.6}$$

with $V^* = \mathrm{E}_x[G_{\tau_1^*(\mathcal{S})}(X_{\tau_1^*(\mathcal{S})})]$. If the collection $\mathfrak{S}$ is rich enough, then

$$\sup_{\mathcal{S} \in \mathfrak{S}} \mathrm{E}_x\bigl[G_{\tau_1(\mathcal{S})}\bigl(X_{\tau_1(\mathcal{S})}\bigr)\bigr] \approx \mathrm{E}_x\bigl[G_{\tau_1(\mathcal{S}^*)}\bigl(X_{\tau_1(\mathcal{S}^*)}\bigr)\bigr]$$

and $V_{M,N}$ can serve as a good approximation for V^* for large enough M and N. In the next section, we will derive some probabilistic bounds for the difference $V^* - V_M$ and show that these bounds are best possible.

3. Main results.

First, we introduce the notion of δ-entropy that plays an important role in the theory of empirical processes. By means of the δ-entropy, the complexity of the class $\mathfrak{S}$ will be measured.

DEFINITION 3.1. Let $\delta > 0$ be a given number and $d_X(\cdot, \cdot)$ be a pseudedistance between two elements of $\mathfrak{B}$ defined as

$$d_X(G_1 \times \cdots \times G_K, G'_1 \times \cdots \times G'_K) = \sum_{k=1}^{K} \mathrm{P}_x\bigl(X(t_k) \in G_k \triangle G'_k\bigr), \tag{3.1}$$

where $\{G_k\}$ and $\{G'_k\}$ are subsets of E. Define $N(\delta, \mathfrak{S}, d_X)$ be the smallest value n for which there exist pairs of sets

$$(G^L_{j,1} \times \cdots \times G^L_{j,K}, G^U_{j,1} \times \cdots \times G^U_{j,K}), \qquad j = 1, \dots, n,$$

such that $d_X(G^L_{j,1} \times \cdots \times G^L_{j,K}, G^U_{j,1} \times \cdots \times G^U_{j,K}) \le \delta$ for all $j = 1, \dots, n$, and for any $G \in \mathfrak{S}$ there exists $j(G) \in \{1, \dots, n\}$ for which

$$G^L_{j(G),k} \subseteq G_k \subseteq G^U_{j(G),k}, \qquad k = 1, \dots, K.$$

Then the value $\mathcal{H}(\delta, \mathfrak{S}, d) := \log[N(\delta, \mathfrak{S}, d_X)]$ is called the δ-entropy with bracketing of $\mathfrak{S}$ for the pseudedistance d_X.

In the sequel, we assume that the δ-entropy with bracketing of the class $\mathfrak{S}$ is polynomial in $1/\delta$. This condition restricts the complexity of the class $\mathfrak{S}$.

ASSUMPTION. We assume that the family of stopping regions $\mathfrak{S}$ is such that

$$\mathcal{H}(\delta, \mathfrak{S}, d_X) \le A\delta^{-\rho} \tag{3.2}$$

for some constant $A > 0$, any $0 < \delta < 1$ and some $\rho > 0$.

The next example shows how to construct a class $\mathfrak{S}$ with the δ-entropy satisfying (3.2).

EXAMPLE 3.2. Let $\mathfrak{S} = \mathfrak{S}_\gamma$, where $\mathfrak{S}_\gamma$ is a class of subsets of $\overbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}^{K}$ with boundaries of Hölder smoothness $\gamma > 0$ defined as follows. For given $\gamma > 0$ and $d \ge 2$, consider the functions $b(x_1, \ldots, x_{d-1}) : \mathbb{R}^{d-1} \to \mathbb{R}$ having continuous partial derivatives of order l, where l is the maximal integer that is strictly less than γ. For such functions b, we denote the Taylor polynomial of order l at a point $x \in \mathbb{R}^{d-1}$ by $\pi_{b,x}$. For a given $H > 0$, let $\Sigma(\gamma, H)$ be the class of functions b such that

$$|b(y) - \pi_{b,x}(y)| \le H\|x - y\|^\gamma, \qquad x, y \in \mathbb{R}^{d-1},$$

where $\|y\|$ stands for the Euclidean norm of $y \in \mathbb{R}^{d-1}$. Any function b from $\Sigma(\gamma, H)$ determines a set

$$S_b := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : 0 \le x_d \le b(x_1, \ldots, x_{d-1})\}.$$

Define the class

$$\mathfrak{S}_\gamma := \{S_{b_1} \times \cdots \times S_{b_{K-1}} \times E : b_1, \ldots, b_{K-1} \in \Sigma(\gamma, H)\}. \tag{3.3}$$

It can be shown [see, Dudley (1999), Section 8.2] that the class $\mathfrak{S}_\gamma$ fulfills

$$\mathcal{H}(\delta, \mathfrak{S}_\gamma, d_X) \le A\delta^{-(K-1)(d-1)/\gamma}$$

for some $A > 0$ and all $\delta > 0$ small enough.

Now we are in the position to formulate our main result that provides exponential bounds for the difference $V^* - V_M$ with V_M given in (2.6).

THEOREM 3.3. *Let $\mathfrak{S}$ be a subset of $\mathfrak{B}$ such that the assumption (3.2) is fulfilled for some ρ satisfying $0 < \rho \le 1$, and*

$$V^* - \bar{V} \le DM^{-1/(1+\rho)} \tag{3.4}$$

with $\bar{V} := \sup_{\mathcal{S} \in \mathfrak{S}} \mathrm{E}_x[G_{\tau_1(\mathcal{S})}(X_{\tau_1(\mathcal{S})})]$ and some constant $D > 0$. Furthermore, assume that all functions G_k are uniformly bounded and the inequalities

$$\mathrm{P}_x\bigl(|G_k(X_k) - \mathrm{E}_k[V^*_{k+1}(X_{k+1})]| \le \delta\bigr) \le A_{0,k}\delta^\alpha, \qquad \delta < \delta_0, \tag{3.5}$$

hold for some $\alpha > 0$, $A_{0,k} > 0$, $k = 1, \dots, K-1$, *and* $\delta_0 > 0$. *Then for any* $U > U_0$ *and* $M > M_0$

$$\mathrm{P}_x^{\otimes M}\bigl(V^* - V_M \geq (U/M)^{(1+\alpha)/(2+\alpha(1+\rho))}\bigr) \leq C \exp\bigl(-\sqrt{U}/B\bigr) \tag{3.6}$$

with some constants $U_0 > 0$, $M_0 > 0$, $B > 0$ *and* $C > 0$.

We stress that the inequality (3.6) has nonasymptotic nature since it holds for all $M > M_0$, where M_0 depends only on the characteristics of the process $(G_k(X_k))_{k>0}$.

REMARK 3.4. Without condition (3.4) the inequality (3.6) continues to hold with V^* replaced by $\bar{V}$, the best approximation of V^* within the class of stopping regions $\mathfrak{S}$.

REMARK 3.5. The requirement that functions G_k are uniformly bounded can be replaced by the existence of all moments of $G_k(X_k)$, $k = 1, \dots, K-1$, under P. In this case, on can reformulate Theorem 7.2 of Section 7 using generalized entropy with bracketing instead of the usual entropy with bracketing [see Chapter 5.4 in Van de Geer (2000)]. We also note that no convexity or smoothness of the functions G_k is required as it usual in the case of stochastic programming problems of the form (1.2).

REMARK 3.6. The choice of the class of approximating sets $\mathfrak{S}$ is very important for a good performance of simulation-based optimization algorithms. On the one hand, if the class $\mathfrak{S}$ is too large, then the optimization over $\mathfrak{S}$ in (2.4) can become infeasible. On the other hand, if $\mathfrak{S}$ is too small, the condition (3.4) may not be fulfilled and the approximation may be too rough. An ingenious choice of $\mathfrak{S}$ should be a trade-off between the above two extremes. In many practical applications it is, however, often clear how to choose a parsimonious parametrization of the stopping regions. This choice can be based on a deep understanding of the nature of the underlying problem or some heuristics (see Section 4 for some examples). An alternative and more constructive way to choose $\mathfrak{S}$ is to use the so called ε-nets. A class of sets $\mathfrak{N} \subset \mathfrak{B}$ is called a ε-net for $\mathfrak{S}$ w.r.t. a pseudo-distance d on $\mathfrak{B}$ if for any $\boldsymbol{\mathcal{S}} \in \mathfrak{S}$ there is $\tilde{\boldsymbol{\mathcal{S}}} \in \mathfrak{N}$ such that $d(\boldsymbol{\mathcal{S}}, \tilde{\boldsymbol{\mathcal{S}}}) \leq \varepsilon$. In the case of distance d defined as the Lebesgue measure of symmetric difference of sets, an ε-net $\mathfrak{N}$ for $\mathfrak{S}$ can be often taken finite. It can be shown that Theorem 3.3 continues to hold if one performs an optimization in (2.4) not over the whole class $\mathfrak{S}$ but only over its ε-net $\mathfrak{N}$, provided that ε tends to 0 with M sufficiently fast.

REMARK 3.7. There is a close connection between the simulation-based optimization algorithm of this paper and the so-called regression-based Monte Carlo approach. The latter one relies on the Wald–Bellman equations (2.2) and tries to approximate all expectations in (2.2) by means of linear or nonlinear regression methods. This approach was first introduced in financial literature on option pricing (see Section 4 for some additional references) and since then become very popular among practitioners. A theoretical analysis of this type of algorithms was done in Clément, Lamberton and Protter (2002), Egloff (2005), Egloff, Kohler and Todorovic (2007) and Kohler, Krzyzak and Todorovic (2010), among others. Both approaches have their advantages and disadvantages. While the simulation-based optimization algorithm requires a careful choice of the class of approximating sets $\mathfrak{S}$ (see Remark 3.6) and involves optimization over $\mathfrak{S}$ that can be rather time consuming, the regression methods are usually fast. On the other hand, for a regression approach to perform well it is necessary to choose a set of basis functions (a bandwidth, a class of sieves) in a proper way. Moreover, the simulation-based optimization approach seems to be rather natural given the structure of the underlying optimal stooping problem (1.1).

REMARK 3.8. The way of estimating the optimal value function V^* presented in Section 2 suggests that one can use the simulation-based optimization algorithm to estimate the boundaries of stopping regions as well. In this case, it would be interesting to reformulate the results of Theorem 3.3 in terms of a distance between $\partial\mathcal{S}^*$ and $\partial\mathcal{S}_M$ which is different from $V^* - V_M$. It is an open problem whether on can relax or completely avoid the conditions (3.4) and (3.5) in this situation.

In order to illustrate the conditions of Theorem 3.3, let us look at a simple example.

EXAMPLE 3.9. Fix some $\alpha > 0$ and $x_0 \in \mathbb{R}_+$ and consider the following optimal stopping problem:

$$V^* = \sup_{\tau \in \{1,2\}} \mathrm{E}[G(X_\tau)|X_0 = x_0], \tag{3.7}$$

where

$$G(x) := (K^{1/\alpha} - x^{1/\alpha})^+, \qquad x \in \mathbb{R}_+, \tag{3.8}$$

with some $K > 0$. Suppose that the Markov chain $(X_k, k = 0, 1, 2)$ originates from the discretization of a continuous process $Y(t)$ which in turn follows the Black–Scholes model with volatility σ and zero interest rate, that is,

$$dY(t) = \sigma Y(t)\,dW(t), \qquad t > 0,\ Y(0) = x_0,$$

and $X_k = Y(k\Delta)$, $k = 0, 1, 2,$ with some $\Delta > 0$. By Itô's formula, the process $Z(t) := Y^{1/\alpha}(t)$ fulfills the following SDE:

$$\frac{dZ(t)}{Z(t)} = \frac{\sigma^2}{2\alpha}\left(\frac{1}{\alpha} - 1\right) dt + \frac{\sigma}{\alpha}\,dW(t).$$

Therefore, the expectation $\mathrm{E}[G(X_2)|X_1 = x]$ can be computed via the well known Black–Scholes formula:

$$\mathrm{E}[G(X_2)|X_1 = x] = K^{1/\alpha}\Phi(-d_2) - x^{1/\alpha}e^{\Delta(\alpha^{-1}-1)(\sigma^2/2\alpha)}\Phi(-d_1), \tag{3.9}$$

with Φ being the cumulative distribution function of the standard normal distribution,

$$d_1 := \frac{\log(x/K) + \sigma^2(\alpha^{-1} - 2^{-1})\Delta}{\sigma\sqrt{\Delta}}$$

and $d_2 := d_1 - \sigma\sqrt{\Delta}/\alpha$. As can be easily seen from (3.9), the function

$$\mathcal{B}(x) := \mathrm{E}[G(X_2)|X_1 = x] - G(x)$$

that appears in (3.5), satisfies $\mathcal{B}(x) \asymp Cx^{1/\alpha}$ as $x \to +0$ for some constant C. Hence,

$$\mathrm{P}\big(0 < |\mathrm{E}[G(X_2)|X_1] - G(X_1)| \le \delta\big) \lesssim \delta^\alpha, \qquad \delta \to 0,\ \alpha > 1,$$

and

$$\mathrm{P}\big(0 < |\mathrm{E}[G(X_2)|X_1] - G(X_1)| \le \delta\big) \lesssim \delta, \qquad \delta \to 0,\ \alpha \le 1.$$

Turn now to the condition (3.4). In fact, for any $\alpha > 0$, the optimal stopping region $\mathcal{S}^* = \{x \in E : \mathcal{B}(x) \le 0\}$ can be represented in the form $\mathcal{S}^* = \{x : 0 \le x \le \theta^*\}$ for some real positive number θ^* depending on α, σ, Δ and K. Hence, if $\mathfrak{S}$ is taken to be a collection of sets of the form $[0, \theta)$ with $\theta \in \Theta \subset \mathbb{R}_+$, we get $\bar{V} = V^*$ and the condition (3.4) is fulfilled.

The convergence rates obtained in Theorem 3.3 are in fact optimal and cannot be, in general, improved as shown in the next theorem.

PROPOSITION 3.10. *Consider the problem* (2.1) *with* $k = 1$ *and two possible stopping dates, that is,* $\tau \in \{1, 2\}$. *Fix a pair of nonzero functions* G_1, G_2 *such that* $G_2 : \mathbb{R}^d \to \{0, 1\}$ *and* $0 < G_1(x) < 1$ *on* $[0, 1]^d$. *Fix some* $\gamma > 0$ *and* $\alpha > 0$ *and let* $\mathcal{P}_{\alpha,\gamma}$ *be a class measures such that the condition* (3.5) *is fulfilled and for any* $\mathrm{P} \in \mathcal{P}_{\alpha,\gamma}$, *the corresponding stopping set* $\boldsymbol{\mathcal{S}}^* = \boldsymbol{\mathcal{S}}^*(\mathrm{P})$ *is in* $\mathfrak{S}_\gamma$. *Then there exist a subset* $\mathcal{P}$ *of* $\mathcal{P}_{\alpha,\gamma}$ *and a constant* $B > 0$ *such that for any* $M \ge 1$, *any stopping time* $\tau_M \in \{1, 2\}$ *measurable w.r.t.* $\mathcal{F}^{\otimes M}$, *it holds*

$$\begin{aligned}\sup_{\mathrm{P}\in\mathcal{P}}\Big\{&\sup_{\tau\in\{1,2\}} \mathrm{E}_{\mathrm{P}}[G_\tau(X_\tau)] - \mathrm{E}_{\mathrm{P}^{\otimes M}}[\mathrm{E}_{\mathrm{P}} G_{\tau_M}(X_{\tau_M})]\Big\}\\ &\ge BM^{-(1+\alpha)/(2+\alpha(1+(d-1)/\gamma))}.\end{aligned}$$

Hence, for any stopping time $\tau_M \in \{1, 2\}$ *measurable w.r.t.* $\mathcal{F}^{\otimes M}$, *there is a measure* P *from* $\mathcal{P}$, *such that*

$$\mathrm{P}^{\otimes M}\big(V^* - V_M \ge CM^{-(1+\alpha)/(2+\alpha(1+(d-1)/\gamma))}\big) > 0 \tag{3.10}$$

with some positive constant C *and all* $M \ge 1$, *where* $V^* = \sup_{\tau\in\{1,2\}} \mathrm{E}_{\mathrm{P}}[G_\tau(X_\tau)]$ *and* $V_M = \mathrm{E}_{\mathrm{P}}[G_{\tau_M}(X_{\tau_M})]$.

REMARK 3.11. In order to compare (3.10) with (3.6) note that $\rho = (d-1)/\gamma$ in the case $\mathfrak{S} = \mathfrak{S}_\gamma$ and $K = 2$ (see Example 3.2).

Discussion. It follows from Theorem 3.3 that

$$V^* - V_M = O_{\mathrm{P}}\big(M^{-(1+\alpha)/(2+\alpha(1+\rho))}\big) = o_{\mathrm{P}}(M^{-1/2})$$

as long as $\alpha > 0$. Using the decomposition

$$V^* - V_{M,N} = V^* - V_M + V_M - V_{M,N}$$

and the fact that $V_M - V_{M,N} = O_{\mathrm{P}}(1/\sqrt{N})$ for any $M > 0$, we conclude that

$$V^* - V_{M,N} = O_{\mathrm{P}}\big(M^{-(1+\alpha)/(2+\alpha(1+\rho))} + N^{-1/2}\big).$$

Hence, given N, a reasonable choice of M, the number of Monte Carlo paths used in the optimization step, can be defined as $M \asymp N^{(2+\alpha(1+\rho))/(2(1+\alpha))}$. In the case when there exists a parametric family of stopping regions satisfying (3.4) (see Example 3.9), one gets

$$M \asymp N^{(2+\alpha)/(2(1+\alpha))} \tag{3.11}$$

since any parametric family of stopping regions with finite-dimensional compact parameter set fulfills (3.2) for arbitrary small $\rho > 0$. Let us also make a few remarks on the condition (3.5) and the parameter α. If all functions

$$\mathcal{B}_k(x) = G_k(x) - \mathrm{E}_{k,x}[V^*_{k+1}(X_{k+1})], \qquad k = 1, \dots, K-1, \tag{3.12}$$

have a nonvanishing Jacobian in the vicinity of the stopping boundary $\partial\mathcal{S}_k$ and X_k has continuous distribution, then (3.5) is fulfilled with $\alpha = 1$. Another situation, where α can be easily determined is described by the following useful lemma.

LEMMA 3.12. *Let $X_1, \dots, X_K$ be a time homogenous Markov chain with a state space $\mathbb{R}_+$ and a transition density $p(y|x) = x^{-1}\bar{p}(y/x)$ such that the function $\bar{p}(z)$ stays positive on $(0,\infty)$ and satisfies $\bar{p}(z) \lesssim z^{-3/2}$, $z \to +\infty$. Moreover, assume that $G_k(x) = a_k(\kappa - x)^+$, where $a_k, k = 1, \dots, K$, is a decreasing sequence of positive numbers and κ is a fixed positive number, then the condition (3.5) is fulfilled with $\alpha \ge 1/2$.*

PROOF. First, note that

$$\mathrm{E}_{K-1,x}[G_K(X_K)] = a_K \int_0^{\kappa/x} (\kappa - zx)\bar{p}(z)\,dz \tag{3.13}$$

and the function

$$\frac{d^2}{dx^2}\mathrm{E}_{K-1,x}[G_K(X_K)] = a_K \frac{\kappa^2}{x^3}\bar{p}(\kappa/x)$$

is positive on $(0,\infty)$. The function $\mathcal{B}_{K-1}(x)$ defined in (3.12) satisfies

$$\mathcal{B}_{K-1}(0) = (a_{K-1} - a_K)\kappa > 0$$

and $\mathcal{B}_{K-1}(x) < 0$ for $x \geq \kappa$. Hence, there is a unique point $x_0 \in (0,\kappa)$ such that $\mathcal{B}_{K-1}(x_0) = 0$. Since $\frac{d^2}{dx^2} G_{K-1}(x) = 0$ on $\mathbb{R}_+ \setminus \{\kappa\}$ and $G_{K-1}(\kappa) = 0$, we get $\mathcal{B}''_{K-1}(x_0) > 0$. Let us now look at the behavior of $\mathcal{B}_{K-1}(x)$ for large x. It directly follows from (3.13) that

$$\mathcal{B}_{K-1}(x) \asymp a_K \bar{p}(+0)\frac{\kappa^2}{2x}, \qquad x \to +\infty.$$

Therefore,

$$\begin{aligned} &\mathrm{P}(|\mathcal{B}_{K-1}(X_{K-1})| \leq \delta) \\ &\quad \leq \mathrm{P}(|X_{K-1} - x_0| \leq A\delta^{1/2}) + \mathrm{P}(X_{K-1} \geq B\delta^{-1}) \lesssim \delta^{1/2}, \qquad \delta \to 0, \end{aligned}$$

for some properly chosen positive constants A and B not depending on δ. In a similar manner, using the fact (it can be proved by induction) that

$$\frac{d^2}{dx^2}\mathrm{E}_{k,x}[V^*_{k+1}(X_{k+1})] > 0, x \in (0,\infty)$$

and $\mathrm{E}_{k,x}[V^*_{k+1}(X_{k+1})] \gtrsim \mathrm{E}_{K-1,x}[G_K(X_K)]$ as $x \to \infty$ for all $k = 1,\dots,K-1$, one derives bounds for other functions $\mathcal{B}_k$, $k = 1,\dots,K-2$. $\square$

In fact, it is not difficult to construct examples showing that the parameter α can take any value from $[1,\infty)$ (see Example 3.9). If $\alpha = 1$ (the most common case) (3.11) simplifies to $M \asymp N^{3/4}$, the rule of thumb supported by our numerical example.

Finally, we would like to mention an interesting methodological connection between our analysis and the analysis of statistical discrimination problem performed in Mammen and Tsybakov (1999) [see also Devroye, Györfi and Lugosi (1996)]. In particular, we need similar results form the theory of empirical processes and the condition (3.5) formally resembles the so-called "margin" condition often encountered in the literature on discrimination analysis.

4. Applications. In this section, we illustrate our theoretical results by some financial applications. Namely, we consider the problem of pricing discrete time American options. According to the modern financial theory, pricing an American option in a complete market is equivalent to solving an optimal stopping problem (with a corresponding generalization in incomplete markets), the optimal stopping time being the rational time for the option to be exercised. Due to the enormous importance of the early exercise feature in finance, this line of research has been intensively pursued in recent times. Solving the optimal stopping problem, and hence pricing an American option is straightforward in low dimensions. However, many problems arising in practice have high dimensions, and these applications have motivated the development of Monte Carlo methods for pricing American option. Solving a high-dimensional optimal stopping problems or pricing American style derivatives with Monte Carlo is a challenging task because the determination of the optimal value function requires a backwards dynamic programming algorithm that appears to be incompatible with the forward nature of Monte Carlo simulation. Much research was focused on the development of fast methods to compute approximations to the optimal value function. Notable examples include mesh method of Broadie and Glasserman (1997), the regression-based approaches of Carriere (1996), Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (1999) and Egloff (2005). All these methods aim at approximating the so-called continuation values that can be used later to construct suboptimal strategies and to produce lower bounds for the optimal value function. The convergence analysis for this type of methods was performed in several papers including Egloff (2005), Egloff, Kohler and Todorovic (2007) and Belomestny (2010). In the context of our paper, we consider the so called parametric approximation algorithms [see, Glasserman (2003), Section 8.2]. In essence, these algorithms represent the optimal stopping sets $\mathcal{S}_k^*$ by a finite numbers of parameters and then find the American option price by maximizing, over the parameter space, a Monte Carlo approximation of the corresponding value function. The important question here is whether on can parametrize the optimal stopping region $\boldsymbol{\mathcal{S}}^*$ by a finite-dimensional set of parameters, i.e. $\boldsymbol{\mathcal{S}}^* = \boldsymbol{\mathcal{S}}^*(\theta), \theta \in \Theta$, where Θ is a compact finite-dimensional set. It turns out that that this is possible in many situations [see Garcia (2003)]. The assumption (3.2) and (3.4) are then automatically fulfilled with arbitrary small $\rho > 0$.

4.1. *Numerical example*: *Bermudan max-call*. This is a benchmark example studied in Broadie and Glasserman (1997) and Glasserman (2003) among others. Specifically, the model with d identically distributed assets is considered, where each underlying has dividend yield δ. The risk-neutral dynamic of the asset $X(t) = (X^1(t), \dots, X^d(t))$ is given by

$$\frac{dX^l(t)}{X^l(t)} = (r - \delta)\,dt + \sigma\,dW^l(t), \qquad X^l(0) = x_0, \qquad l = 1, \dots, d,$$

where $W^l(t), l = 1, \dots, d$, are independent one-dimensional Brownian motions and x_0, r, δ, σ are constants. At any time $t \in \{t_1, \dots, t_K\}$ the holder of the option may exercise it and receive the payoff

$$G_k(X_k) := \bigl(\max(X_k^1, \dots, X_k^d) - \kappa\bigr)^+,$$

where $X_k := X(t_k)$ for $k = 1, \dots, K$. We take $d = 2$, $r = 5\%$, $\delta = 10\%$, $\sigma = 0.2$, $\kappa = 100$, $x_0 = 90$ and $t_k = kT/K, k = 1, \dots, K$, with $T = 3, K = 9$ as in Glasserman (2003), Chapter 8.

To describe the optimal early exercise region at date t_k, $k = 1, \dots, K$, one can divide $\mathbb{R}^2$ into three different connected sets: one exercise region and two continuation regions [see Broadie and Detemple (1997) for more details]. All these regions can be parameterized by using two functions depending on two-dimensional parameter $\theta_k \in \mathbb{R}^2$. Making use of this characterization, we define a parametric family of stopping regions as in Garcia (2003) via

$$\mathcal{S}_k(\theta_k) := \{(x_1, x_2) : \max(\max(x_1, x_2) - K, 0) > \theta_k^1; |x_1 - x_2| > \theta_k^2\},$$

where $\theta_k \in \Theta$, $k = 1, \dots, K$ and Θ is a compact subset of $\mathbb{R}^2$. Furthermore, we simplify the corresponding optimization problem by setting $\theta_1 = \cdots = \theta_K$. This will introduce an additional bias and hence may increase the left-hand side of (3.4) (see Remark 3.4). However, this bias turns out to be rather small in practice. In order to implement and analyze the simulation-based optimization based algorithm in this situation, we perform the following steps:

- Simulate L independent sets of trajectories of the process (X_k) each of the size M:

  $$(X_1^{(l,m)}, \dots, X_K^{(l,m)}), \qquad m = 1, \dots, M,$$

  where $l = 1, \dots, L$.
- Compute estimates $\theta_M^{(1)}, \dots, \theta_M^{(L)}$ via

  $$\theta_M^{(l)} := \arg\max_{\theta \in \Theta} \left\{ \frac{1}{M} \sum_{m=1}^{M} G_{\tau_1(\boldsymbol{\mathcal{S}}(\theta))}(X^{(l,m)}_{\tau_1(\boldsymbol{\mathcal{S}}(\theta))}) \right\}.$$

  To compute estimates $\theta_M^{(1)}, \dots, \theta_M^{(L)}$, we use Tom Rowan's subspace-searching simplex algorithm for unconstrained maximization of a function (package `subplex` in `R`). This choice of optimization algorithm responds to the discontinuity of the value function, together with the presence of multiple local maxima.
- Simulate a new set of trajectories of size N independent of $(X_k^{(l,m)})$:

  $$(\widetilde{X}_1^{(n)}, \dots, \widetilde{X}_K^{(n)}), \qquad n = 1, \dots, N.$$

- Compute L estimates for the optimal value function V_1^* as follows

  $$V_{M,N}^{(l)} := \frac{1}{N} \sum_{n=1}^{N} G_{\tau_M^{(l,n)}}(\widetilde{X}^{(n)}_{\tau_M^{(l,n)}}), \qquad l = 1, \dots, L,$$

  with

  $$\tau_M^{(l,n)} := \min\{1 \le k \le K : \widetilde{X}_k^{(n)} \in \mathcal{S}_k(\theta_M^{(l)})\}, \qquad n = 1, \dots, N.$$

  Denote by $\sigma_{M,N,l}$ the standard deviation computed from the sample $(G_{\tau_M^{(l,n)}}, n = 1, \dots, N)$ and set $\sigma_{M,N} = \min_l \sigma_{M,N,l}$.

- Compute

$$\mu_{M,N,L} := \frac{1}{L}\sum_{l=1}^{L} V_{M,N}^{(l)}, \qquad \vartheta_{M,N,L} := \sqrt{\frac{1}{L-1}\sum_{l=1}^{L}\bigl(V_{M,N}^{(l)} - \mu_{M,N,L}\bigr)^2}.$$

By the law of large numbers

$$\mu_{M,N,L} \xrightarrow{\mathrm{P}} \mathrm{E}_{\mathrm{P}\otimes\mathcal{M}}[V_{M,N}], \qquad L \to \infty, \tag{4.1}$$

$$\vartheta_{M,N,L} \xrightarrow{\mathrm{P}} \mathrm{Var}_{\mathrm{P}\otimes\mathcal{M}}[V_{M,N}], \qquad L \to \infty, \tag{4.2}$$

where

$$V_{M,N} := \frac{1}{N}\sum_{n=1}^{N} G_{\tau_M^{(n)}}\bigl(\widetilde{X}^{(n)}_{\tau_M^{(n)}}\bigr).$$

The difference $\bar{V} - V_{M,N}$ with

$$\bar{V} := \max_{\theta\in\Theta} \mathrm{E}\bigl[G_{\tau_1(\mathcal{S}(\theta))}\bigl(X_{\tau_1(\mathcal{S}(\theta))}\bigr)\bigr]$$

can be decomposed into the sum of three terms

$$(\bar{V} - \mathrm{E}_{\mathrm{P}\otimes\mathcal{M}}[V_M]) + (\mathrm{E}_{\mathrm{P}\otimes\mathcal{M}}[V_M] - V_M) + V_M - V_{M,N}. \tag{4.3}$$

The first term in (4.3) is deterministic and can be approximated by $Q_1(M) := \mu_{M^*,N^*,L^*} - \mu_{M,N^*,L^*}$ with large enough L^*, M^* and N^*. The variability of the second, zero mean, stochastic term can be measured by $\sqrt{\mathrm{Var}_{\mathrm{P}\otimes\mathcal{M}}[V_M]}$ which in turn can be estimated by $Q_2(M) := \sqrt{\vartheta_{M,N^*,L^*}}$, due to (4.2). The standard deviation of $V_M - V_{M,N}$ for any M can be approximated by $Q_3(N) = \sigma_{M^*,N}/\sqrt{N}$. In our simulation study, we take $N^* = 1{,}000{,}000$, $L^* = 500$, $M^* = 10{,}000$ and obtain $\bar{V} \approx \mu_{M^*,N^*,L^*} = 7.96$ [note that $V^* = 8.07$ according to Glasserman (2003)]. In the left-hand side of Figure 1, we plot both quantities $Q_1(M)$ and $Q_2(M)$ as functions of M. Note that $Q_2(M)$ dominates $Q_1(M)$, especially for large M. Hence, by comparing $Q_2(M)$ with $Q_3(N)$ and approximately solving the equation $Q_2(M) = Q_3(N)$ in N, one can infer on the optimal relation between M and N. In Figure 1 (on the right-hand side), the resulting empirical relation is depicted by crosses. Additionally, we plotted two benchmark curves $N = M^{4/3}$ and $N = M^{4.5/3}$. As one can see the choice $M = N^{3/4}$ is likely to be sufficient in this situation since it always leads to the inequality $Q_1(M) + \sigma Q_2(M) \le \sigma Q_3(N)$ for any $\sigma > 1$. As a consequence, for $M = N^{3/4}$ and any N, $\bar{V}$ lies with high probability in the interval $[\mu_{M,N,L^*} - \sigma Q_3(N), \mu_{M,N,L^*} + \sigma Q_3(N)]$, provided that σ is large enough.

5. Proofs of the main results. In this section, we give the proofs of Theorem 3.3 and Proposition 3.10.

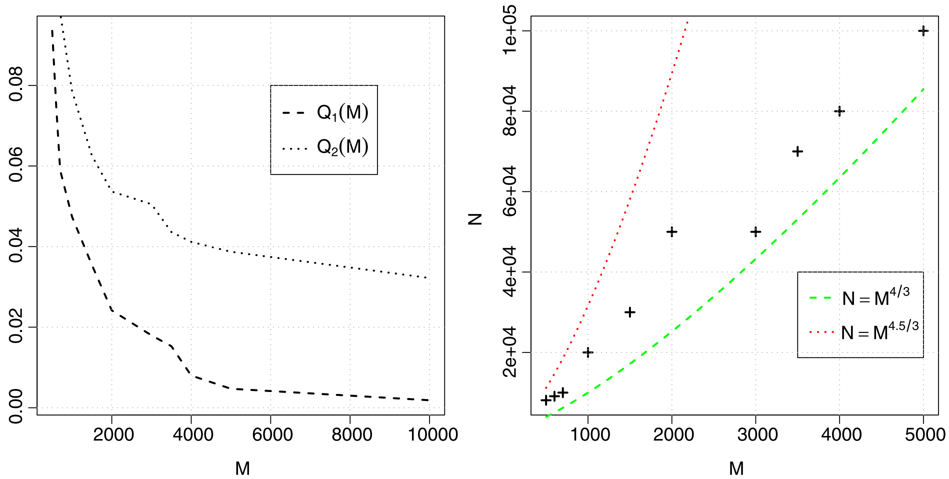

FIG. 1. *Left: functions $Q_1(M)$ and $Q_2(M)$; right: optimal empirical relationship between M and N (crosses) together with benchmark curves $N = M^{4/3}$ (dashed line) and $N = M^{4.5/3}$ (dotted line).*

5.1. *Proof of Theorem* 3.3. Let us first sketch the structure of the proof and main ideas behind it. For any $\boldsymbol{\mathcal{S}} \in \mathfrak{S}$, denote

$$\Delta(\boldsymbol{\mathcal{S}}) := \mathrm{E}[G_{\tau_1^*}(X_{\tau_1^*})] - \mathrm{E}[G_{\tau_1(\boldsymbol{\mathcal{S}})}(X_{\tau_1(\boldsymbol{\mathcal{S}})})].$$

To prove Theorem 3.3, we need a kind of probabilistic bound for the quantity $\Delta(\boldsymbol{\mathcal{S}}_M)$ with $\boldsymbol{\mathcal{S}}_M$ defined in (2.4). In a first step, we separate a probabilistic error from an approximation error. The latter one can be quantified by the value $\Delta(\bar{\boldsymbol{\mathcal{S}}})$, where

$$\bar{\boldsymbol{\mathcal{S}}} := \arg\max_{\boldsymbol{\mathcal{S}} \in \mathfrak{S}} \mathrm{E}[G_{\tau_1(\boldsymbol{\mathcal{S}})}(X^{(m)}_{\tau_1(\boldsymbol{\mathcal{S}})})] \tag{5.1}$$

is the best approximation of $\mathrm{E}[G_{\tau_1^*}(X_{\tau_1^*})]$ within the class of stopping regions $\mathfrak{S}$. Define now

$$\Delta_M(\boldsymbol{\mathcal{S}}) := M^{-1/2} \sum_{m=1}^{M} \{G_{\tau_1(\boldsymbol{\mathcal{S}})}(X^{(m)}_{\tau_1(\boldsymbol{\mathcal{S}})}) - \mathrm{E}[G_{\tau_1(\boldsymbol{\mathcal{S}})}(X_{\tau_1(\boldsymbol{\mathcal{S}})})]\}$$

and put $\Delta_M(\boldsymbol{\mathcal{S}}', \boldsymbol{\mathcal{S}}) := \Delta_M(\boldsymbol{\mathcal{S}}') - \Delta_M(\boldsymbol{\mathcal{S}})$ for any $\boldsymbol{\mathcal{S}}', \boldsymbol{\mathcal{S}} \in \mathfrak{S}$. The empirical process $\Delta_M(\boldsymbol{\mathcal{S}}', \boldsymbol{\mathcal{S}})$ defined on $\mathfrak{B} \times \mathfrak{B}$ shall play a crucial role in obtaining a probabilistic bound for $\Delta(\bar{\boldsymbol{\mathcal{S}}})$. Indeed, since

$$\frac{1}{M} \sum_{m=1}^{M} G_{\tau_1(\bar{\boldsymbol{\mathcal{S}}})}(X^{(m)}_{\tau_1(\bar{\boldsymbol{\mathcal{S}}})}) \le \frac{1}{M} \sum_{m=1}^{M} G_{\tau_1(\boldsymbol{\mathcal{S}}_M)}(X^{(m)}_{\tau_1(\boldsymbol{\mathcal{S}}_M)})$$

with probability 1, it holds

$$\Delta(\boldsymbol{\mathcal{S}}_M) \le \Delta(\bar{\boldsymbol{\mathcal{S}}}) + \frac{[\Delta_M(\boldsymbol{\mathcal{S}}^*, \bar{\boldsymbol{\mathcal{S}}}) + \Delta_M(\boldsymbol{\mathcal{S}}_M, \boldsymbol{\mathcal{S}}^*)]}{\sqrt{M}}. \tag{5.2}$$

Thus, in order to get a bound for $\Delta(\boldsymbol{\mathcal{S}}_M)$ we need probabilistic bounds for the quantities $\Delta_M(\boldsymbol{\mathcal{S}}^*, \bar{\boldsymbol{\mathcal{S}}})$ and $\Delta_M(\boldsymbol{\mathcal{S}}_M, \boldsymbol{\mathcal{S}}^*)$. These bounds in turn can be derived from the exponential inequalities for the increments of empirical processes which are stated in Theorem 7.2 (see Section 7). Let us elaborate on this point in more detail. Set $\varepsilon_M = M^{-1/2(1+\rho)}$ and derive from (5.2)

$$\Delta(\boldsymbol{\mathcal{S}}_M) \le \Delta(\bar{\boldsymbol{\mathcal{S}}}) + \frac{2}{\sqrt{M}} \sup_{\boldsymbol{\mathcal{S}}\in\mathfrak{S}:\Delta_G(\boldsymbol{\mathcal{S}}^*,\boldsymbol{\mathcal{S}})\le\varepsilon_M} |\Delta_M(\boldsymbol{\mathcal{S}}^*, \boldsymbol{\mathcal{S}})| \tag{5.3}$$
$$+ 2 \times \frac{\Delta_G^{(1-\rho)}(\boldsymbol{\mathcal{S}}^*, \boldsymbol{\mathcal{S}}_M)}{\sqrt{M}} \sup_{\boldsymbol{\mathcal{S}}\in\mathfrak{S}:\Delta_G(\boldsymbol{\mathcal{S}}^*,\boldsymbol{\mathcal{S}})>\varepsilon_M} \left[\frac{|\Delta_M(\boldsymbol{\mathcal{S}}^*, \boldsymbol{\mathcal{S}})|}{\Delta_G^{(1-\rho)}(\boldsymbol{\mathcal{S}}^*, \boldsymbol{\mathcal{S}})}\right],$$

where

$$\Delta_G(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}') := \{\mathrm{E}[G_{\tau_1(\boldsymbol{\mathcal{S}})}(X_{\tau_1(\boldsymbol{\mathcal{S}})}) - G_{\tau_1(\boldsymbol{\mathcal{S}}')}(X_{\tau_1(\boldsymbol{\mathcal{S}}')})]^2\}^{1/2}$$

for any $\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}' \in \mathfrak{B}$. The reason behind splitting the right-hand side of (5.2) into two parts is that the behavior of the empirical process $\Delta_M(\boldsymbol{\mathcal{S}}^*, \boldsymbol{\mathcal{S}})$ is different on the sets $\{\boldsymbol{\mathcal{S}} \in \mathfrak{S} : \Delta_G(\boldsymbol{\mathcal{S}}^*, \boldsymbol{\mathcal{S}}) > \varepsilon_M\}$ and $\{\boldsymbol{\mathcal{S}} \in \mathfrak{S} : \Delta_G(\boldsymbol{\mathcal{S}}^*, \boldsymbol{\mathcal{S}}) \le \varepsilon_M\}$. Theorem 7.2 of Section 7 would imply that for any $\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}' \in \mathfrak{S}$ and any $U > U_0$

$$\mathrm{P}\Big(\sup_{\boldsymbol{\mathcal{S}}'\in\mathfrak{S},\Delta_G(\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{S}}')\le\varepsilon_M} |\Delta_M(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')| > U\varepsilon_M^{1-\rho}\Big) \le C\exp(-U\varepsilon_M^{-2\rho}/C^2), \tag{5.4}$$

$$\mathrm{P}\left(\sup_{\boldsymbol{\mathcal{S}}'\in\mathfrak{S},\Delta_G(\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{S}}')>\varepsilon_M} \frac{|\Delta_M(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')|}{\Delta_G^{1-\rho}(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')} > U\right) \le C\exp(-U/C^2), \tag{5.5}$$

$$\mathrm{P}\Big(\sup_{\boldsymbol{\mathcal{S}}\in\mathfrak{S}} |\Delta_M(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')| > z\sqrt{M}\Big) \le C\exp(-Mz^2/C^2B) \tag{5.6}$$

with some constants $C > 0$, $B > 0$ and $U_0 > 0$, provided that

$$\mathcal{H}_B(\delta, \mathfrak{S}, \Delta_G) \le A\delta^{-2\rho}, \tag{5.7}$$

where $\mathcal{H}_B(\delta, \mathfrak{S}, \Delta_G)$ is the entropy with bracketing for the class $\mathfrak{S}$ w.r.t. the pseudo-distance Δ_G. The condition (5.7) places a bound on the complexity of $\mathfrak{S}$ and is similar to (3.2). However, in order to deduce (5.7) from (3.2) we need to relate the pseudo-distance Δ_G to the pseudo-distance d_X defined in (3.1). The following lemma relates Δ_G to another auxiliary pseudo-distance and is proved in Section 6.

LEMMA 5.1. *If* $\max_{k=1,\dots,K} \|G_k\|_\infty \le A_G$ *with some constant* $A_G > 0$, *then*

$$\Delta_G(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}') \le 2A_G\sqrt{K\,\Delta_X(\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}')}$$

for any $\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}' \in \mathfrak{B}$, *where* Δ_X *is a pseudo-distance between any two sets* $\boldsymbol{\mathcal{S}}, \boldsymbol{\mathcal{S}}' \in \mathfrak{B}$ *defined as*

$$\Delta_X(\mathcal{S}_1 \times \cdots \times \mathcal{S}_K, \mathcal{S}_1' \times \cdots \times \mathcal{S}_K') := \sum_{k=1}^{K-1} \mathrm{P}\left(X_k \in (\mathcal{S}_k \triangle \mathcal{S}_k') \setminus \left(\bigcap_{l=k}^{K-1} \mathcal{S}_l'\right)\right).$$

In fact, Lemma 5.1 and the assumption (3.2) immediately imply (5.7) since $\Delta_X(\mathcal{S},\mathcal{S}')\le d_X(\mathcal{S},\mathcal{S}')$. So the inequalities (5.4)–(5.6) hold under assumptions of Theorem 3.3. Let us now show how these inequalities can be used to estimate the second and the third summands in (5.3). To simplify notations, denote

$$\mathcal{W}_{1,M}:=\sup_{\mathcal{S}\in\mathfrak{S}:\Delta_G(\mathcal{S}^*,\mathcal{S})\le\varepsilon_M}|\Delta_M(\mathcal{S}^*,\mathcal{S})|,$$

$$\mathcal{W}_{2,M}:=\sup_{\mathcal{S}\in\mathfrak{S}:\Delta_G(\mathcal{S}^*,\mathcal{S})>\varepsilon_M}\frac{|\Delta_M(\mathcal{S}^*,\mathcal{S})|}{\Delta_G^{(1-\rho)}(\mathcal{S}^*,\mathcal{S})}$$

and set $\mathcal{A}_0:=\{\mathcal{W}_{1,M}\le U\varepsilon_M^{1-\rho}\}$ for some $U>U_0$. Then the inequality (5.4) leads to the estimate

$$\mathrm{P}(\bar{\mathcal{A}}_0)\le C\exp(-U\varepsilon_M^{-2\rho}/C^2).$$

Furthermore, since $\Delta(\bar{\mathcal{S}})\le DM^{-1/(1+\rho)}$ [see (3.4)] and $\varepsilon_M^{1-\rho}/\sqrt{M}=M^{-1/(1+\rho)}$, we get on $\mathcal{A}_0$

$$\Delta(\mathcal{S}_M)\le C_0M^{-1/(1+\rho)}+2\times\frac{\Delta_G^{(1-\rho)}(\mathcal{S}^*,\mathcal{S}_M)}{\sqrt{M}}\mathcal{W}_{2,M}\tag{5.8}$$

with $C_0=D+2U$. Now we need to find a bound for $\Delta_G(\mathcal{S}^*,\mathcal{S}_M)$ in terms of $\Delta(\mathcal{S}_M)$. This is exactly the place, where the condition (3.5) comes in. The following lemma holds.

LEMMA 5.2. *Assume that* (3.5) *holds for* $\delta<\delta_0<1/2$, *then there exist constants* υ_α *and* δ_α *such that*

$$\Delta(\mathcal{S})\ge\upsilon_\alpha\Delta_X^{(1+\alpha)/\alpha}(\mathcal{S}^*,\mathcal{S})\tag{5.9}$$

for all $\mathcal{S}\in\mathfrak{B}$ *satisfying* $\Delta_X(\mathcal{S}^*,\mathcal{S})\le\delta_\alpha$. *Moreover, it holds*

$$\Delta_X(\mathcal{S}^*,\mathcal{S})\le\left(\frac{2^{1/\alpha}}{\delta_0}\right)\Delta(\mathcal{S})+\frac{\delta_\alpha}{2(1+\alpha)}\tag{5.10}$$

for any $\mathcal{S}\in\mathfrak{B}$.

The proof of this lemma is given in Section 6. Lemma 5.2 together with Lemma 5.1 imply now that

$$\Delta_G(\mathcal{S}^*,\mathcal{S}_M)\le 2\sqrt{K}A_G\upsilon_\alpha^{-\alpha/2(1+\alpha)}\Delta^{\alpha/2(1+\alpha)}(\mathcal{S}_M)\tag{5.11}$$

on the set $\mathcal{A}_1:=\{\Delta_X(\mathcal{S}^*,\mathcal{S}_M)\le\delta_\alpha\}$. Let us introduce yet another set

$$\mathcal{A}_2:=\{\Delta(\mathcal{S}_M)>C_0(1-\pi)^{-1}M^{-1/(1+\rho)}\}$$

for some $0<\pi<1$. Combining (5.8) with (5.11), we get on $\mathcal{A}_0\cap\mathcal{A}_1\cap\mathcal{A}_2$

$$\Delta(\mathcal{S}_M)\le C_1\frac{\Delta^{\alpha(1-\rho)/(2(1+\alpha))}(\mathcal{S}_M)}{\pi\sqrt{M}}\mathcal{W}_{2,M},$$

where the constant C_1 depends on α but not on π. Therefore,

$$\Delta(\boldsymbol{\mathcal{S}}_M) \le (\pi/C_1)^{-\nu} M^{-\nu/2} \mathcal{W}^{\nu}_{2,M}$$

with $\nu = \frac{2(1+\alpha)}{2+\alpha(1+\rho)}$. What remains is to estimate $\mathrm{P}(\bar{\mathcal{A}}_1)$. Using again Lemma 5.2, we arrive at

$$\begin{aligned}\mathrm{P}\big(\Delta_X(\boldsymbol{\mathcal{S}}^*, \boldsymbol{\mathcal{S}}_M) > \delta_\alpha\big) &\le \mathrm{P}\biggl(\biggl(\frac{2^{1/\alpha}}{\delta_0}\biggr)\Delta(\boldsymbol{\mathcal{S}}_M) + \frac{\delta_\alpha}{2(1+\alpha)} > \delta_\alpha\biggr)\\ &= \mathrm{P}\big(\Delta(\boldsymbol{\mathcal{S}}_M) > c_\alpha\big)\end{aligned}$$

with $c_\alpha = \delta_0\delta_\alpha 2^{-1/\alpha}(1 - \frac{1}{2(1+\alpha)})$. Furthermore, due to (5.2)

$$\begin{aligned}\mathrm{P}\big(\Delta(\boldsymbol{\mathcal{S}}_M) > c_\alpha\big) &\le \mathrm{P}\Bigl(DM^{-1/(1+\rho)} + 2M^{-1/2} \sup_{\boldsymbol{\mathcal{S}}\in\mathfrak{S}} |\Delta_M(\boldsymbol{\mathcal{S}})| > c_\alpha\Bigr)\\ &\le \mathrm{P}\Bigl(\sup_{\boldsymbol{\mathcal{S}}\in\mathfrak{S}} |\Delta_M(\boldsymbol{\mathcal{S}})| > c_\alpha\sqrt{M}/4\Bigr)\end{aligned}$$

for large enough M. In order to bound the latter probability, we can employ the inequality (5.6) to get

$$\mathrm{P}\Bigl(\sup_{\boldsymbol{\mathcal{S}}\in\mathfrak{S}} |\Delta_M(\boldsymbol{\mathcal{S}})| > c_\alpha\sqrt{M}/4\Bigr) \le B_1 \exp(-MB_2)$$

with some constants $B_1 > 0$ and $B_2 = B_2(\alpha) > 0$. Thus,

$$\mathrm{P}(\bar{\mathcal{A}}_1) \le B_1 \exp(-MB_2).$$

Applying inequality (5.5) to $\mathcal{W}^{\nu}_{2,M}$ and using the fact that $\nu/2 \le 1/(1+\rho)$ for all $0 < \rho \le 1$, we finally obtain the desired bound for $\Delta(\boldsymbol{\mathcal{S}}_M)$

$$\begin{aligned}&\mathrm{P}\big(\Delta(\boldsymbol{\mathcal{S}}_M) > (V/M)^{\nu/2}\big)\\ &\quad\le \mathrm{P}\big(\{\Delta(\boldsymbol{\mathcal{S}}_M) > (V/M)^{\nu/2}\} \cap \mathcal{A}_0 \cap \mathcal{A}_1 \cap \mathcal{A}_2\big) + \mathrm{P}(\bar{\mathcal{A}}_0) + \mathrm{P}(\bar{\mathcal{A}}_1)\\ &\quad\le C\exp\big(-\sqrt{V}/B_3\big) + C\exp\biggl(-\frac{UM^{\rho/(1+\rho)}}{C^2}\biggr) + B_1\exp(-MB_2)\end{aligned}$$

which holds for all $V > V_0$ and $M > M_0$ with some constant B_3 depending on π and α.

5.2. *Proof of Proposition* 3.10. For simplicity, we give the proof only for the case $d = 2$ (an extension to higher dimensions is straightforward). In the case of two exercise dates, the corresponding optimal stopping problem is completely specified by the distribution of the vector $(X_1, G_2(X_2))$. Because of a digital structure of G_2, the distribution of $(X_1, G_2(X_2))$ would be completely determined if the marginal distribution of X_1 and the probability $\mathrm{P}(G_2(X_2) = 1|X_1 = x)$ are defined. Taking into account this, we now construct a family of distributions for

$(X_1, G_2(X_2))$ indexed by elements of the set $\Omega = \{0,1\}^m$. First, the marginal distribution of X_1 is supposed to be the same for all $\omega \in \Omega$ and possesses a density $p(x)$ satisfying

$$0 < p_* \le p(x) \le p^* < \infty, \qquad x \in [0,1]^2.$$

Let us now construct a family of conditional distributions $\mathrm{P}_\omega(G_2(X_2) = 1|X_1 = x)$, $\omega \in \Omega$. To this end, let ϕ be an infinitely many times differentiable function on $\mathbb{R}$ with the following properties: $\phi(z) = 0$ for $|z| \ge 1$, $\phi(z) \ge 0$ for all z and $\sup_{z\in\mathbb{R}}[\phi(z)] \le 1$. For $j = 1, \dots, m$, put

$$\phi_j(z) := \delta m^{-\gamma}\phi\biggl(m\biggl[z - \frac{2j-1}{m}\biggr]\biggr), \qquad z \in \mathbb{R},$$

with some $0 < \delta < 1$. For vectors $\omega = (\omega_1, \dots, \omega_m)$ of elements $\omega_j \in \{0,1\}$ and for any $z \in \mathbb{R}$, define

$$b(z,\omega) := \sum_{j=1}^{m} \omega_j \phi_j(z).$$

Put for any $\omega \in \Omega$ and any $x \in \mathbb{R}^2$,

$$\begin{aligned} C_\omega(x) &:= \mathrm{P}_\omega\bigl(G_2(X_2) = 1|X_1 = x\bigr) \\ &= G_1(x) - Am^{-\gamma/\alpha}\mathbf{1}\{0 \le x_2 \le b(x_1,\omega)\} \\ &\quad + Am^{-\gamma/\alpha}\mathbf{1}\{b(x_1,\omega) < x_2 \le \delta m^{-\gamma}\}, \end{aligned}$$

where A is a positive constant. Due to our assumptions on $G_1(x)$, there are constants $0 < G_- < G_+ < 1$ such that

$$G_- \le G_1(x) \le G_+, \qquad x \in [0,1]^2.$$

Hence, the constant A can be chosen in such a way that $C_\omega(x)$ remains positive and strictly less than 1 on $[0,1]^2$ for any $\omega \in \Omega$. The stopping set

$$\mathcal{S}_\omega := \{x : C_\omega(x) \le G_1(x)\} = \{(x_1,x_2) : 0 \le x_2 \le b(x_1,\omega)\}$$

belongs to $\mathfrak{S}_\gamma$ since $b(\cdot,\omega) \in \Sigma(\gamma, L)$ for δ small enough. Moreover, for any $\eta > 0$

$$\begin{aligned} \mathrm{P}_\omega\bigl(|G_1(X_1) - C_\omega(X_1)| \le \eta\bigr) &= \mathrm{P}_\omega(0 \le X_1^2 \le \delta m^{-\gamma})\mathbf{1}(Am^{-\gamma/\alpha} \le \eta) \\ &\le \delta p^* m^{-\gamma}\mathbf{1}(Am^{-\gamma/\alpha} \le \eta) \le \delta p^* A^{-\alpha}\eta^\alpha \end{aligned}$$

and the condition (3.5) is fulfilled. Let τ_M be a stopping time w.r.t. $\mathcal{F}^{\otimes M}$, then the identity (see Lemma 6.1)

$$\begin{aligned} &\mathrm{E}_{\mathrm{P}_\omega}[G_{\tau^*}(X_{\tau^*})] - \mathrm{E}_{\mathrm{P}_\omega}[G_{\tau_M}(X_{\tau_M})] \\ &\quad = \mathrm{E}_{\mathrm{P}_\omega}\bigl[\bigl(G_1(X_1) - G_2(X_2)\bigr)\mathbf{1}(\tau^* = 1, \tau_M = 2)\bigr] \\ &\qquad + \mathrm{E}_{\mathrm{P}_\omega}\bigl[\bigl(G_2(X_2) - G_1(X_1)\bigr)\mathbf{1}(\tau^* = 2, \tau_M = 1)\bigr] \\ &\quad = \mathrm{E}_{\mathrm{P}_\omega}[|G_1(X_1) - \mathrm{E}(G_2(X_2)|\mathcal{F}_1)|\mathbf{1}\{\tau_M \ne \tau^*\}] \end{aligned}$$

leads to

$$\mathrm{E}_{\mathrm{P}_\omega}[G_{\tau^*}(X_{\tau^*})] - \mathrm{E}_{\mathrm{P}_\omega^{\otimes M}}\{\mathrm{E}_{\mathrm{P}_\omega}[G_{\tau_M}(X_{\tau_M})]\} = \mathrm{E}_{\mathrm{P}_\omega^{\otimes M}}\mathrm{E}_{\mathrm{P}_\omega}[|\Delta_\omega(X_1)|\mathbf{1}\{\tau_M \neq \tau^*\}]$$

with $\Delta_\omega(x) := G_1(x) - C_\omega(x)$. By conditioning on X_1, we get

$$\begin{aligned}\mathrm{E}_{\mathrm{P}^{\otimes M}}\mathrm{E}_{\mathrm{P}_\omega}[|\Delta_\omega(X_1)|\mathbf{1}\{\tau_M \neq \tau^*\}] &= Am^{-\gamma/\alpha}\mathrm{P}(0 \le X_1^2 \le \delta m^{-\gamma})\mathrm{P}_\omega^{\otimes M}(\tau_M \neq \tau^*) \\ &\ge Am^{-\gamma/\alpha} p_* \delta m^{-\gamma} \mathrm{P}_\omega^{\otimes M}(\tau_M \neq \tau^*).\end{aligned}$$

Using now a well-known Birgé's or Huber's lemma [see, e.g., Devroye, Györfi and Lugosi (1996), page 243], we get

$$\sup_{\omega\in\{0,1\}^m} \mathrm{P}_\omega^{\otimes M}(\widehat{\tau}_M \neq \tau^*) \ge \left[0.36 \wedge \left(1 - \frac{MK_{\mathcal{H}}}{\log(|\mathcal{H}|)}\right)\right],$$

where $K_{\mathcal{H}} := \sup_{P,Q\in\mathcal{H}} K(P,Q)$, $\mathcal{H} := \{\mathrm{P}_\omega, \omega \in \{0,1\}^m\}$ and $K(P,Q)$ is a Kullback–Leibler distance between two measures P and Q. Since for any two measures P and Q from $\mathcal{H}$ with $Q \neq P$

$$\begin{aligned}K(P,Q) \le \sup_{\substack{\omega_1,\omega_2\in\{0,1\}^m\\ \omega_1\neq\omega_2}} \mathrm{E}\biggl[&C_{\omega_1}(X_1)\log\left\{\frac{C_{\omega_1}(X_1)}{C_{\omega_2}(X_1)}\right\} \\ &+ \bigl(1 - C_{\omega_1}(X_1)\bigr)\log\left\{\frac{1 - C_{\omega_1}(X_1)}{1 - C_{\omega_2}(X_1)}\right\}\biggr] \\ &\le (1 - G_+ - A)^{-1}(G_- - A)^{-1}\mathrm{P}(0 \le X_1^2 \le \delta m^{-\gamma})[A^2 m^{-2\gamma/\alpha}] \\ &\le CMm^{-\gamma-2\gamma/\alpha-1}\end{aligned}$$

with some constant $C > 0$ for small enough A, and $\log(|\mathcal{H}|) = m\log(2)$, we get

$$\sup_{\omega\in\{0,1\}^m} \mathrm{P}_\omega^{\otimes M}(\widehat{\tau}_M \neq \tau^*) \ge [0.36 \wedge (1 - CMm^{-\gamma-2\gamma/\alpha-1})]$$

with some constant $C > 0$. Hence,

$$\sup_{\omega\in\{0,1\}^m} \mathrm{P}_\omega^{\otimes M}(\widehat{\tau}_M \neq \tau^*) > 0$$

provided that $m = qM^{1/(\gamma+2\gamma/\alpha+1)}$ for small enough real number $q > 0$. In this case,

$$\begin{aligned}&\sup_{\omega\in\{0,1\}^m} \bigl\{\mathrm{E}_{\mathrm{P}_\omega}[G_{\tau^*}(X_{\tau^*})] - \mathrm{E}_{\mathrm{P}_\omega^{\otimes M}}\{\mathrm{E}_{\mathrm{P}_\omega}[G_{\tau_M}(X_{\tau_M})]\}\bigr\} \\ &\qquad \ge Ap_*\delta q^{-\gamma/\alpha-\gamma} M^{-(\gamma/\alpha+\gamma)/(\gamma+2\gamma/\alpha+1)} = BM^{-(1+\alpha)/(2+\alpha(1+1/\gamma))}\end{aligned}$$

with $B = Ap_*\delta q^{-\gamma/\alpha-\gamma}$.

6. Proofs of lemmas. In this section, we prove Lemmas 5.1 and 5.2. The proofs of both lemmas essentially rely on the following proposition.

PROPOSITION 6.1. *For any $\mathcal{S}, \mathcal{S}' \in \mathfrak{B}$, it holds with probability one*

$$\begin{aligned} &\big|G_{\tau_k(\mathcal{S})}\big(X_{\tau_k(\mathcal{S})}\big) - G_{\tau_k(\mathcal{S}')}\big(X_{\tau_k(\mathcal{S}')}\big)\big| \\ &\quad \le \sum_{l=k}^{K-1} \big|G_l(X_l) - G_{\tau_{l+1}(\mathcal{S})}\big(X_{\tau_{l+1}(\mathcal{S})}\big)\big| \mathbf{1}_{\{X_l \in (\mathcal{S}_l \triangle \mathcal{S}'_l) \setminus (\bigcap_{l'=l}^{K-1} \mathcal{S}'_{l'})\}} \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} &V_k^*(X_k) - V_k(X_k) \\ &\quad = \mathrm{E}\Bigg[\sum_{l=k}^{K-1} \big|G_l(X_l) - \mathrm{E}[V_{l+1}^*(X_{l+1})|\mathcal{F}_l]\big| \mathbf{1}_{\{X_l \in (\mathcal{S}_l^* \triangle \mathcal{S}_l) \setminus (\bigcap_{l'=l}^{K-1} \mathcal{S}_{l'})\}} \Big| \mathcal{F}_k\Bigg] \end{aligned} \tag{6.2}$$

for $k = 1, \dots, K-1$, where

$$V_k(X_k) := \mathrm{E}\big[G_{\tau_k(\mathcal{S})}\big(X_{\tau_k(\mathcal{S})}\big)|\mathcal{F}_k\big], \qquad k = 1, \dots, K.$$

Before proving this proposition let us recall some basic properties of the sequence of stopping times $\tau_k(\mathcal{S}), k = 1, \dots, K$, with $\mathcal{S} \in \mathfrak{S}$. First, it immediately follows from the definition of τ_k that $\tau_k(\mathcal{S}) = k$ iff $X_k \in \mathcal{S}_k$, $k = 1, \dots, K$. In particular, $\tau_K(\mathcal{S}) = K$ with probability 1. Next, the sequence $\tau_k(\mathcal{S})$ satisfies the so-called consistency property

$$\text{if } X_k \notin \mathcal{S}_k \text{ then } \tau_k(\mathcal{S}) = \tau_{k+1}(\mathcal{S}), \qquad k = 1, \dots, K-1.$$

Let us also recall that due to the Wald–Bellman equation (2.2)

$$V_k^*(X_k) = \begin{cases} \mathrm{E}[V_{k+1}^*(X_{k+1})|\mathcal{F}_k], & X_k \notin \mathcal{S}_k^*, \\ G_k(X_k), & X_k \in \mathcal{S}_k^* \end{cases}$$

for $k = 1, \dots, K-1$.

PROOF. We prove (6.2) by induction. The inequality (6.1) can be proved in a similar way. For $k = K-1$, we get

$$\begin{aligned} &V_{K-1}^*(X_{K-1}) - V_{K-1}(X_{K-1}) \\ &\quad = \mathrm{E}\big[\big(G_{K-1}(X_{K-1}) - G_K(X_K)\big)\mathbf{1}_{\{X_{K-1} \in \mathcal{S}_{K-1}^*, X_{K-1} \notin \mathcal{S}_{K-1}\}}|\mathcal{F}_{K-1}\big] \\ &\qquad + \mathrm{E}\big[\big(G_K(X_K) - G_{K-1}(X_{K-1})\big)\mathbf{1}_{\{X_{K-1} \notin \mathcal{S}_{K-1}^*, X_{K-1} \in \mathcal{S}_{K-1}\}}|\mathcal{F}_{K-1}\big] \\ &\quad = |G_{K-1}(X_{K-1}) - \mathrm{E}[G_K(X_K)|\mathcal{F}_{K-1}]|\mathbf{1}_{\{X_{K-1} \in \mathcal{S}_{K-1}^* \triangle \mathcal{S}_{K-1}\}} \end{aligned}$$

since the events $\{X_{K-1} \notin \mathcal{S}_{K-1}^*\}$ and $\{X_{K-1} \notin \mathcal{S}_{K-1}\}$ are measurable w.r.t. $\mathcal{F}_{K-1}$ and $G_{K-1}(X_{K-1}) \ge \mathrm{E}[G_K(X_K)|\mathcal{F}_{K-1}]$ on the set $\{X_{K-1} \in \mathcal{S}_{K-1}^*\}$. Thus, (6.2) holds with $k = K - 1$. Suppose that (6.2) holds with $k = K' + 1$. Let us prove it for $k = K'$. Consider a decomposition

$$G_{\tau^*_{K'}}(X_{\tau^*_{K'}}) - G_{\tau_{K'}}(X_{\tau_{K'}}) = S_1 + S_2 + S_3 \tag{6.3}$$

with

$$S_1 := \big(G_{\tau^*_{K'}}(X_{\tau^*_{K'}}) - G_{\tau_{K'}}(X_{\tau_{K'}})\big)\mathbf{1}_{\{X_{K'}\notin\mathcal{S}^*_{K'}, X_{K'}\notin\mathcal{S}_{K'}\}},$$
$$S_2 := \big(G_{\tau^*_{K'}}(X_{\tau^*_{K'}}) - G_{\tau_{K'}}(X_{\tau_{K'}})\big)\mathbf{1}_{\{X_{K'}\notin\mathcal{S}^*_{K'}, X_{K'}\in\mathcal{S}_{K'}\}},$$
$$S_3 := \big(G_{\tau^*_{K'}}(X_{\tau^*_{K'}}) - G_{\tau_{K'}}(X_{\tau_{K'}})\big)\mathbf{1}_{\{X_{K'}\in\mathcal{S}^*_{K'}, X_{K'}\notin\mathcal{S}_{K'}\}}.$$

Using the fact that $\tau_k = \tau_{k+1}$ if $X_k \notin \mathcal{S}_k$ for any $k = 1, \dots, K-1$, we get

$$\begin{aligned}
\mathrm{E}[S_1|\mathcal{F}_{K'}] &= \mathrm{E}\big[\big(V^*_{K'+1}(X_{K'+1}) - V_{K'+1}(X_{K'+1})\big)\mathbf{1}_{\{X_{K'}\notin\mathcal{S}^*_{K'}, X_{K'}\notin\mathcal{S}_{K'}\}}|\mathcal{F}_{K'}\big],\\
\mathrm{E}[S_2|\mathcal{F}_{K'}] &= \big(\mathrm{E}[G_{\tau^*_{K'+1}}(X_{\tau^*_{K'+1}})|\mathcal{F}_{K'}] - G_{K'}(X_{K'})\big)\mathbf{1}_{\{X_{K'}\notin\mathcal{S}^*_{K'}, X_{K'}\in\mathcal{S}_{K'}\}}\\
&= \big(\mathrm{E}[V^*_{K'+1}(X_{K'+1})|\mathcal{F}_{K'}] - G_{K'}(X_{K'})\big)\mathbf{1}_{\{X_{K'}\notin\mathcal{S}^*_{K'}, X_{K'}\in\mathcal{S}_{K'}\}}
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{E}[S_3|\mathcal{F}_{K'}]\\
&\quad= \big(G_{K'}(X_{K'}) - \mathrm{E}[G_{\tau_{K'+1}}(X_{\tau_{K'+1}})|\mathcal{F}_{K'}]\big)\mathbf{1}_{\{X_{K'}\in\mathcal{S}^*_{K'}, X_{K'}\notin\mathcal{S}_{K'}\}}\\
&\quad= \big(G_{K'}(X_{K'}) - \mathrm{E}[V^*_{K'+1}(X_{K'+1})|\mathcal{F}_{K'}]\big)\mathbf{1}_{\{X_{K'}\in\mathcal{S}^*_{K'}, X_{K'}\notin\mathcal{S}_{K'}\}}\\
&\qquad+ \mathrm{E}\big[\big(V^*_{K'+1}(X_{K'+1}) - V_{K'+1}(X_{K'+1})\big)\mathbf{1}_{\{X_{K'}\in\mathcal{S}^*_{K'}, X_{K'}\notin\mathcal{S}_{K'}\}}|\mathcal{F}_{K'}\big],
\end{aligned}$$

with probability one. Hence,

$$\begin{aligned}
&V^*_{K'}(X_{K'}) - V_{K'}(X_{K'})\\
&\quad= |G_{K'}(X_{K'}) - \mathrm{E}[V^*_{K'+1}(X_{K'+1})|\mathcal{F}_{K'}]|\mathbf{1}_{\{X_{K'}\in\mathcal{S}^*_{K'}\triangle\mathcal{S}_{K'}\}}\\
&\qquad+ \mathrm{E}\big[\big(V^*_{K'+1}(X_{K'+1}) - V_{K'+1}(X_{K'+1})\big)|\mathcal{F}_{K'}\big]\mathbf{1}_{\{X_{K'}\notin\mathcal{S}_{K'}\}}
\end{aligned}$$

since $G_{K'}(X_{K'}) - \mathrm{E}[V^*_{K'+1}(X_{K'+1})|\mathcal{F}_{K'}] \ge 0$ on the set $\{X_{K'} \in \mathcal{S}^*_{K'}\}$ and $G_{K'}(X_{K'}) - \mathrm{E}[V^*_{K'+1}(X_{K'+1})|\mathcal{F}_{K'}] \le 0$ on the set $\{X_{K'} \notin \mathcal{S}^*_{K'}\}$ [see (2.3)]. Our induction assumption implies now that

$$\begin{aligned}
&V^*_{K'}(X_{K'}) - V_{K'}(X_{K'})\\
&\quad= \mathrm{E}\left[\sum_{l=K'}^{K-1} |G_l(X_l) - \mathrm{E}[V^*_{l+1}(X_{l+1})|\mathcal{F}_l]|\mathbf{1}_{\{X_l\in(\mathcal{S}^*_l\triangle\mathcal{S}_l)\setminus(\bigcap_{l'=l}^{K-1}\mathcal{S}_{l'})\}}\middle|\mathcal{F}_{K'}\right]
\end{aligned}$$

and hence (6.2) holds with $k = K'$. $\square$

Let us turn now to the proof of Lemma 5.1. We get by (6.1)

$$\begin{aligned}\Delta_G(\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{S}}') &= \{\mathrm{E}[G_{\tau_1(\boldsymbol{\mathcal{S}})}(X_{\tau_1(\boldsymbol{\mathcal{S}})}) - G_{\tau_1(\boldsymbol{\mathcal{S}}')}(X_{\tau_1(\boldsymbol{\mathcal{S}}')})]^2\}^{1/2}\\ &\le 2A_G\sqrt{\mathrm{E}\Biggl[\sum_{l=1}^{K-1}\mathbf{1}_{\{X_l\in(\mathcal{S}_l\triangle\mathcal{S}'_l)\setminus(\bigcap_{l'=l}^{K-1}\mathcal{S}'_{l'})\}}\Biggr]^2}\\ &\le 2A_G\sqrt{K\sum_{l=1}^{K-1}\mathrm{P}\Biggl\{X_l\in(\mathcal{S}_l\triangle\mathcal{S}'_l)\Bigm\backslash\Biggl(\bigcap_{l'=l}^{K-1}\mathcal{S}'_{l'}\Biggr)\Biggr\}}\\ &= 2A_G\sqrt{K\Delta_X(\boldsymbol{\mathcal{S}},\boldsymbol{\mathcal{S}}')}.\end{aligned}$$

The proof of Lemma 5.2 is a little bit more involved and relies on the assumption (3.5). For any $\delta \le \delta_0$, define the sets

$$\mathcal{A}_k := \{x\in\mathbb{R}^d : |\mathrm{E}[V^*_{k+1}(X_{k+1})|X_k = x] - G_k(x)| > \delta\}, \qquad k = 1,\dots,K-1.$$

Due to (6.2), we have

$$\begin{aligned}\Delta(\boldsymbol{\mathcal{S}}) &\ge \delta\sum_{k=1}^{K-1}\mathrm{P}\Biggl(X_k\in(\mathcal{S}^*_k\triangle\mathcal{S}_k)\Bigm\backslash\Biggl(\bigcap_{l=k}^{K-1}\mathcal{S}_k\Biggr)\cap\mathcal{A}_k\Biggr)\\ &\ge \delta\sum_{k=1}^{K-1}\Biggl\{\mathrm{P}\Biggl(X_k\in(\mathcal{S}^*_k\triangle\mathcal{S}_k)\Bigm\backslash\Biggl(\bigcap_{l=k}^{K-1}\mathcal{S}_k\Biggr)\Biggr) - \mathrm{P}(\bar{\mathcal{A}}_k)\Biggr\} \qquad (6.4)\\ &\ge \delta[\Delta_X(\boldsymbol{\mathcal{S}}^*,\boldsymbol{\mathcal{S}}) - A_0\delta^\alpha]\end{aligned}$$

with $A_0 = \sum_{k=1}^{K-1} A_{k,0}$, where $A_{k,0}$ were defined in (3.5). The maximum of (6.4) is attained at $\delta^* = [\Delta_X(\boldsymbol{\mathcal{S}}^*,\boldsymbol{\mathcal{S}})/(\alpha+1)A_0]^{1/\alpha}$. Since $\delta^* \le \delta_0$ for $\Delta_X(\boldsymbol{\mathcal{S}}^*,\boldsymbol{\mathcal{S}}) \le A_0(\alpha+1)\delta_0^\alpha$, the inequality (5.9) holds with $\upsilon_\alpha := A_0^{-1/\alpha}\alpha(1+\alpha)^{-1-1/\alpha}$ and $\delta_\alpha := A_0(\alpha+1)\delta_0^\alpha$. The inequality (5.10) directly follows from (6.4) by taking $\delta = \delta_0/2^{1/\alpha}$.

7. Exponential inequalities for the increments of empirical processes. In this section, we will use the notation introduced in Section 2. In particular, let $X_1,\dots,X_K$ be a Markov chain with the joint distribution P_X and let

$$(X_1^{(m)},\dots,X_K^{(m)}), \qquad m = 1,\dots,M,$$

be M independent copies of X. For any set $\boldsymbol{\mathcal{S}} \in \mathfrak{B}$, define the empirical process $\nu_M(\boldsymbol{\mathcal{S}})$ via

$$\begin{aligned}\nu_M(\boldsymbol{\mathcal{S}}) &:= M^{-1/2}\sum_{m=1}^{M}\{g_{\boldsymbol{\mathcal{S}}}(X_1^{(m)},\dots,X_K^{(m)}) - \mathrm{E}[g_{\boldsymbol{\mathcal{S}}}(X_1,\dots,X_K)]\}\\ &= \sqrt{M}\int g_{\boldsymbol{\mathcal{S}}}\,d(P_X^{\otimes M} - P_X)\end{aligned}$$

with functions $g_{\mathcal{S}} : \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{K} \to \mathbb{R}$ defined as

$$g_{\mathcal{S}}(x_1, \dots, x_K) := \sum_{k=0}^{K-1} G_{k+1}(x_{k+1}) \mathbf{1}_{\{x_1 \notin \mathcal{S}_1, \dots, x_k \notin \mathcal{S}_k, x_{k+1} \in \mathcal{S}_{k+1}\}}.$$

Denote $\mathcal{G} = \{g_{\mathcal{S}} : \mathcal{S} \in \mathfrak{S}\}$ and define the entropy with bracketing of the class $\mathcal{G}$.

DEFINITION 7.1. Let $\mathcal{N}_B(\delta, \mathcal{G}, \mathrm{P}_X)$ be the smallest value of n for which there exist pairs of functions $\{[g_j^L, g_j^U]\}_{j=1}^n$ such that $\|g_j^U - g_j^L\|_{L_2(\mathrm{P}_X)} \le \delta$ for all $j = 1, \dots, n$, and such that for each $g \in \mathcal{G}$, there is $j = j(g) \in \{1, \dots, n\}$ such that

$$g_j^L \le g \le g_j^U.$$

Then $\mathcal{H}_B(\delta, \mathcal{G}, \mathrm{P}_X) = \log[\mathcal{N}_B(\delta, \mathcal{G}, \mathrm{P}_X)]$ is called the entropy with bracketing of $\mathcal{G}$.

The following theorem provides us with the exponential bounds for the increment $\nu_M(\mathcal{S}) - \nu_M(\mathcal{S}_0)$, where $\mathcal{S}_0$ is a fixed element of $\mathfrak{S}$.

THEOREM 7.2. *Assume that there exists a constant $A > 0$ such that*

$$\mathcal{H}_B(\delta, \mathcal{G}, \mathrm{P}_X) \le A\delta^{-\varkappa} \tag{7.1}$$

for any $\delta > 0$ and some $\varkappa > 0$, where $\mathcal{H}_B(\delta, \mathcal{G}, \mathrm{P}_X)$ is the δ-entropy with bracketing of $\mathcal{G}$. Fix some $\mathcal{S}_0 \in \mathfrak{S}$ then for $\varepsilon = M^{-1/(2+\varkappa)}$ the following inequalities hold

$$\mathrm{P}\Big(\sup_{\mathcal{S} \in \mathfrak{S}, \|g_{\mathcal{S}} - g_{\mathcal{S}_0}\|_{L_2(\mathrm{P}_X)} \le \varepsilon} |\nu_M(\mathcal{S}) - \nu_M(\mathcal{S}_0)| > U\varepsilon^{1-\varkappa/2}\Big) \le C \exp(-U\varepsilon^{-\varkappa}/C^2),$$

$$\mathrm{P}\Big(\sup_{\mathcal{S} \in \mathfrak{S}, \|g_{\mathcal{S}} - g_{\mathcal{S}_0}\|_{L_2(\mathrm{P}_X)} > \varepsilon} \frac{|\nu_M(\mathcal{S}) - \nu_M(\mathcal{S}_0)|}{\|g_{\mathcal{S}} - g_{\mathcal{S}_0}\|_{L_2(\mathrm{P}_X)}^{1-\varkappa/2}} > U\Big) \le C \exp(-U/C^2)$$

for all $U > C$ and $M > M_0$, where C and M_0 are two positive constants. Moreover, for any $z > 0$

$$\mathrm{P}\Big(\sup_{\mathcal{S} \in \mathfrak{S}} |\nu_M(\mathcal{S}) - \nu_M(\mathcal{S}_0)| > z\sqrt{M}\Big) \le C \exp(-Mz^2/C^2 B)$$

with some positive constant $B > 0$.

Theorem 7.2 follows from Theorems 5.11 and 5.13 in Van de Geer (2000). Let us make this statement more precise. First, note that $\mathcal{G}$ is a uniformly bounded class of functions provided that all functions G_k are uniformly bounded. The first inequality of Theorem 7.2 follows from the inequality (5.42) of Lemma 5.13 in Van de Geer (2000) if we take $\beta = 0$, $\alpha = \varkappa$. Similarly, the second inequality is a direct consequence of the inequality (5.43) of the same Lemma 5.13. Finally, the third inequality of Theorem 7.2 can be derived from the inequality (5.35) of Theorem 5.11 in Van de Geer (2000) by taking $a = L\sqrt{n}$ with small enough, but independent of n, constant L [see also the proof of Theorem 5.13 in Van de Geer (2000)].

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