

## Derived Hecke algebra and Venkatesh's conjectures

Summer term 2018

The goal of this mini-seminar is to partially understand Venkatesh's derived Hecke algebra and its action on the cohomology of locally symmetric spaces. In the first two talks we review basic facts about arithmetic manifolds and Hecke algebras. Talks three to five deal with the abstract structure of the derived Hecke algebra. In the last talk, we try to understand the action of the Hecke algebra in the first non-trivial example. This is already surprisingly hard and involves the description of the cohomology in question in terms of algebraic  $K$ -theory.

**Talk 1. Locally symmetric spaces** The aim of this talk is to give some background on arithmetic locally symmetric spaces. In the seminar we use the adelic language. First, we want to see how to move from the classical language to the adelic one in the case of  $GL_n$ . (see Theorem 3.3.1 and Proposition 3.3.1 of [Bum98] and the book of Humphreys [Hum80]). The second task is to show that arithmetic groups are virtually torsion-free, i.e. contain a torsion-free normal subgroup of finite index. An elementary proof of this fact can be found in [Alp87]. Therefore, locally symmetric spaces can be considered as classifying spaces (see Chapter 6.10 of [Wei95].)

**Talk 2. Classical Hecke algebras and Satake isomorphism** Define local Hecke algebras and their actions on cohomology groups. Explain the Satake isomorphism for spherical Hecke algebras (see Section 1,2 and 3 of [Gro98]).

**Talk 3. Derived Hecke algebra I** Discuss the different but equivalent definitions of the local derived Hecke algebra and construct its action on cohomology (see Section 2.1-2.5 and Appendix A of [Ven16]).

**Talk 4. Derived Hecke algebra II** Explain how the abstract action of the derived Hecke algebra on the cohomology of an arithmetic manifold can be made explicit (Section 2.6-2.10 and Appendix A of [Ven16]). Define the global derived Hecke algebra (Section 2.11 and 2.12 of [Ven16]).

**Talk 5. Derived Satake isomorphism** Prove the derived Satake isomorphism (see Section 3 of [Ven16]) for spherical derived Hecke algebras. As an important corollary we get that (under some technical hypotheses) spherical derived Hecke algebras are graded-commutative.

**Talk 6. Trivial part of the cohomology** Let  $G$  be the group of units of a division algebra over an imaginary quadratic field. The aim of this talk is to show that the "trivial part" of the cohomology of the associated locally symmetric space is cyclic over the derived Hecke algebra (Theorem 5.2 of [Ven16]).

### REFERENCES

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