The derived Hecke algebra for dihedral weight one forms

Our goal is to understand the recent preprint [DRHV21] by Darmon, Harris, Rotger and Venkatesh. In the preprint the authors verify some instances of a conjecture by and Harris–Venkatesh [HV19] on the action of the derived Hecke algebra on modular forms of weight one. More precisely, to an eigenform g corresponds a 2-dimensional Galois representation ρ_g, and the conjecture predicts the existence of certain units u_g in the splitting field of the adjoint of ρ_g that can be used to describe the action of the derived Hecke operators. The preprint [DRHV21] proves the existence of the relevant unit u_g when g is dihedral; that is, when ρ_g is induced from a ray class character of a quadratic extension of $\mathbb{Q}$. The proof relies on results of Gross [Gro87] and Emerton [Eme02] on an integral version of the Jacquet-Langlands correspondence for definite quaternion algebras, as well as on the more recent notion of higher Eisenstein elements introduced by Merel [Mer96] and Lecouturier [Lec21].

Roughly half of the talks cover background material on elliptic curves, modular curves/forms and theta series. In the second half of the seminar we go through the preprint [DRHV21] as well as some background material from the paper [Eme02]. There are two very different cases one has to distinguish: either the quadratic extension from which the Galois representation is induced is real quadratic or it is imaginary quadratic. We stick to the imaginary quadratic case.

Talk 1. Introduction (07.04.2022) Explain the conjecture of [HV19], the main theorem of [DRHV21] and give an overview over the seminar.

Talk 2. Elliptic curves (14.04.2022) This talk covers the basics on elliptic curves. An elliptic curve over a field k is a smooth projective curve E of genus one together with rational point $O \in E(k)$. It turns out that these curve naturally carry the structure of an algebraic group.

- Recall the Riemann-Roch theorem for curves ([Sil86], Section II.5)
- Show that every elliptic curve is given by a Weierstraß equation and every such equation gives an elliptic curve ([Sil86], Proposition III.3.1)
- Sketch the geometric construction of the group law ([Sil86], Section III.2)
- Show that the map

  $$E \longrightarrow \mathrm{Pic}^0(E),\ P \mapsto [P] - [0]$$

  is an isomorphism ([Sil86],Proposition III.3.4)

Talk 3. Isogenies (21.04.2022) Show that an isogeny ϕ is a group homomorphism and has finite kernel ([Sil86], Section III.4). Define its degree, the dual isogeny $\hat{\phi}$ and prove that $\phi \circ \hat{\phi} = \deg(\phi)$ ([Sil86], Section III.6, in particular Theorem III.6.2). Deduce the structure of the m-torsion subgroup of an elliptic curve ([Sil86], Corollary III.6.4)

Talk 4. Supersingular elliptic curves (28.04.2022) Give the definition of a supersingular elliptic curve over $\overline{\mathbb{F}_N}$ ($N > 3$ prime). Explain why all supersingular elliptic curves are isogenous. Define the definite quaternion algebra B_N over $\mathbb{Q}$ ramified at N and explain that the endomorphism algebra of a supersingular elliptic curve is isomorphic to B_N. Recall some basic definitions in the theory of quaternion algebras: (oriented) maximal orders, ideals and the Picard group. Discuss the statement at the bottom of page 8 of [DRHV21] giving a bijection between $\mathrm{Pic}(B_N)$ and the set of isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}_N}$.

References: Most of the above is explained in detail in chapter 42 of [Voi21], [Vig80], and Chapter V.3 of [Sil86].

Talk 5. Analytic theory of modular curves/forms (05.05.2022) Sketch the equivalence between elliptic curves over $\mathbb{C}$ and complex one-dimensional tori (see [Sil86], Section VI). Deduce that the complex upper half space modulo the action of $\mathrm{SL}_2(\mathbb{Z})$ parametrizes complex elliptic curves ([Dei13], Theorem 2.1.5). Give similar descriptions for quotients by congruence subgroups $\Gamma_0(N)$ and $\Gamma_1(N)$. Explain how one can compactify these spaces by adding cusps.

Define modular forms, cusp forms and their q-expansion (see [Dei13] or [DS05]). Mention that modular forms of weight 2 are sections of the canonical line bundle on the completed modular curve. Explain the action of the Hecke algebra and give formulas in terms of the q-expansion. And most importantly, give lots of examples; in particular, Eisenstein series of level one, the Δ-function, Theta series of even-dimensional definite quadratic spaces ([Iwa97], Theorem 10.8), Theta series attached to imaginary quadratic fields ([Iwa97], Theorem 12.5)

Talk 6. Algebraic theory of modular curves/forms (12.05.2022) Explain the moduli problems corresponding to the group $\Gamma_0(N)$ and $\Gamma_1(N)$. Sketch the proof that the $\Gamma_1(N)$-moduli problem is representable by the modular curve $Y_1(N)$ and that it is smooth over $\mathbb{Z}[1/N]$ (see Proposition 3.3.2, Proposition 3.3.5, Definition 3.3.6 and Proposition 3.4.3 of [Loe]). Construct $Y_0(N)$ as a quotient of $Y_1(N)$ and state that it is smooth and a coarse moduli space (see Definition 3.6.2 of *loc.cit.*). Briefly explain how to compactify modular curves via moduli of generalized elliptic curves as in [DR73].

Recast modular forms in terms of sections of line bundles on the modular curve and give the moduli theoretic description of Hecke operators (see for example Section 3.1, 4.1 and 4.3 of [Fü]). Compare with the construction of the derived Hecke operator in [HV19].

Talk 7. Explicit Jacquet-Langlands correspondence (19.05.2022) Let $N > 3$ be a prime and let $\mathbb{T}(N)$ be the Hecke algebra acting on modular forms of level $\Gamma_0(N)$. Let $\mathcal{M}$ denote the $\mathbb{Z}$-module of weight two modular forms for $\Gamma_0(N)$ with $a_n(f) \in \mathbb{Z}$ for $n \geq 1$ and $2a_0(f) \in \mathbb{Z}$. Let $\mathcal{X}$ be the free abelian group on the set of supersingular elliptic curves over $\overline{\mathbb{F}_N}$.

The goal of talks 3 and 4 is to discuss the proofs of some of the theorems in [Eme02] concerning $\mathcal{M}$ and $\mathcal{X}$, particularly Theorem 3.1, Theorem 4.2 and Theorem 0.3.

- Explain the interpretation of $\mathcal{X}$ as the group of divisors on the geometric fibre of $X_0(N)$ over $\overline{\mathbb{F}_N}$ supported on singular points.
- Define the algebra $\mathbb{T}(N)$ and the notion of an Eisenstein ideal. Then describe the action of $\mathbb{T}(N)$ on $\mathcal{X}$ and explain the proof that $\mathcal{X}$ is a faithful $\mathbb{T}(N)$–module ([Eme02], Theorem 3.1]. Recall the main properties of Néron models used in the proof.
- Discuss the other results at the beginning of Section 3 of [Eme02] up to Proposition 3.15. Explain how they lead to the pairing (3.10) of [Eme02] and the explicit pairing $\Theta\colon \mathcal{X} \otimes_{\mathbb{T}(N)} \mathcal{X} \longrightarrow \mathcal{M}$ (see [DRHV21], (16)]).

Talk 8. Hecke modules and the Eisenstein ideal (02.06.2022) This is a continuation of the previous talk. The goal is to study the module $\mathcal{X}$ "near" an Eisenstein ideal $\mathfrak{m}$ of $\mathbb{T}(N)$. It should discuss the proof of [Eme02], Theorem 4.2, and [Eme02], Theorem 0.3, and explain how this implies that the map

$$\mathcal{X}_{\mathfrak{m}} \otimes_{\mathbb{T}(N)_{\mathfrak{m}}} \mathcal{X}_{\mathfrak{m}} \longrightarrow \mathcal{M}_{\mathfrak{m}}$$

is an isomorphism of $\mathbb{T}(N)_{\mathfrak{m}}$-modules. It would be good to discuss the example $N = 11$ (see [Eme02], pp. 6-8) to illustrate the main results of this and the previous talk.

Talk 9. A trace identity for definite theta series (9.06.2022) This talk discusses the proof of [DRHV21], Theorem 2.2. Thus, one should cover most of Section 2, which is more or less self-contained. We always assume that the occurring integer D is prime. In particular, one can skip Section 2.6.

Talk 10. Higher Eisenstein elements 1 (23.06.2022)
The goal of talks 6 and 7 is to cover [DRHV21], Section 4. This section introduces the notion of higher Eisenstein elements due to Merel and Lecouturier and gives examples in several Hecke modules. The two talks combined should cover:

- Sections 4.1 and 4.2 of [DRHV21] leading up to Definition 4.6. The proof of Proposition 4.1 should be given in detail.
- Theorem 4.9, which characterizes the Shimura class as a higher Eisenstein element.
- Theorem 4.11, which gives an explicit description of a higher Eisenstein element in $\mathcal{X}$. This is one of the most original results in the paper, so the second talk should go over the proof in detail.

Talk 11. Higher Eisenstein elements 2 (30.06.2022)
Continuation of the previous talk.

Talk 12. Proof of the main theorem (in the definite case) (07.07.2022)
This talk should cover Section 5.1 and 5.2 of [DRHV21], which use all the previous material to give the proof of the main Theorem 1.2 of *loc.cit.* in the definite case.

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