Shalika models and p-adic L-function

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Structure of talk

1. Main Theorem
2. Period integrals
3. P-adic interpolation
**F** totally real number field

*V* = \( \otimes V_v \) cuspidal automorphic representation of \( GL_{2n} / F \) such that

- \( V \) is cohomological with respect to \( V_{al} \)
- \( V \) has a Shalika model

Examples:

- \( f \) modular form of weight \( k \geq 2 \) - \( V_{al} = Sym^{k-2} \mathbb{C}^2 \)
- \( \sim \) \( Sym^3 f \) "symmetric cube"
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$F_\infty$ maximal abelian unramified outside $p$ and $\infty$ extension of $F$

$G_p = \text{Gal}(F_\infty, F)$

$s$ critical (half-)integer of $V$

Aim: $p$-adically interpolate $L(V \otimes \chi, s)$ for

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\chi : G_p \rightarrow \mathbb{C}^*
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More precisely: Construct $p$-adic measure $\mu$ on $G_p$ such that

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- $V$ admits a weak ordinary $p$-stabilization $\Theta$
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Then such a measure $\mu$ exists.

Ash-Ginzburg 1994, Dimitrov-Januszewski-Raghuram 2018
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<table>
<thead>
<tr>
<th></th>
<th>Main Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Period integrals</td>
</tr>
<tr>
<td>3</td>
<td>P-adic interpolation</td>
</tr>
</tbody>
</table>

Shalika models and p-adic L-function
Modular forms

\[ L(f, 1) = -2\pi i \int_0^{i\infty} f(z)dz \]

First observation:
cycle from 0 to \( \infty \) is orbit of maximal torus in \( GL_2(\mathbb{R}) \)
Second observation:
If \( k = 2 \), then
\[ f \in H^1_c(\Gamma \backslash \mathbb{H}, \mathbb{C}) \]
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\[ H = \text{GL}_n \times \text{GL}_n \subset G = \text{GL}_n \]
\[ \hookrightarrow \text{inclusion of symmetric spaces } X_H \hookrightarrow X_G \]
\[ \hookrightarrow \text{cycle } [X_H] \text{ of dimension } q \text{ on } \Gamma \backslash X_G \]

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Group cohomology

Remember:

\[ H^i(\Gamma \backslash X_G, \mathbb{C}) = H^i(\Gamma, \mathbb{C}) \]

What about cohomology with compact support?

Borel-Serre compactification yields

\[ H^i_c(\Gamma \backslash X_G, \mathbb{C}) = H^{i-l}(\Gamma, \text{Hom}(D_G, \mathbb{C})) \]

\( D_G \) "Steinberg module" - free \( \mathbb{Z} \)-module of infinite rank

Example: \( G = GL_2/F \leadsto D_G = \text{Div}_0(\mathbb{P}^1(F)), \ i = 1 \)
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$V_p$ local component of $V$ at $p$ (defined over number field $E$) 
$\sim$ modified Eichler-Shimura map 
$\sim$ classes $\omega \in H^{q-l}(\Gamma(p), \text{Hom}(D_G, \text{Hom}(V_p, \mathbb{C})))$

$\Gamma(p) \subseteq \text{PGL}_{2n}(\mathcal{O}[1/p])$ - $p$-arithmetic group

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compatible with classical Eichler-Shimura map
Rationality

**Lemma**

*The canonical map*

\[ H^i(\Gamma(p), \text{Hom}(D_G, \text{Hom}(V_p, E))) \otimes \mathbb{C} \]

\[ \rightarrow H^i(\Gamma(p), \text{Hom}(D_G, \text{Hom}(V_p, \mathbb{C}))) \]

*is an isomorphism and the space is finite-dimensional.*

*Idea of proof: Schneider-Stuhler resolution*

\[ 0 \rightarrow \text{c-ind}_{K_0}^{G_p} L_0 \rightarrow \cdots \rightarrow \text{c-ind}_{K_r}^{G_p} L_r \rightarrow V_p \rightarrow 0 \]

with \( L_j \) finite-dimensional

*Shapiro’s Lemma \( \Rightarrow \) reduce to finite-dimensional coefficients*
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Main Theorem
Period integrals
P-adic interpolation

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What about integrality?

Want integral model $\mathcal{L}$ of $V_p$ (over $\mathbb{Z}$ or $\mathbb{Z}_p$) with resolution as before - "cohomologically integral"

$\Rightarrow$ get integral structure on cohomology

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The map delta I

Idea: construct a map

\[ \delta: \text{function space} \rightarrow V_p \]

Pullback to \( H \) yields

\[ \delta^\vee: H^{q-I}(\Gamma^{(p)}, \text{Hom}(D_G, \text{Hom}(V_p, E))) \]
\[ \rightarrow H^{q-I}(\Gamma^{(p)}_H, \text{Hom}(D_H, \text{distributions})) \]

Pushforward along determinant gives \( \mu \)

"weakly ordinary" \( \leadsto \delta \) respects integral models
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Assumption: $V_p$ has a stabilization, i.e., a non-trivial map

$$\Theta: \text{Ind}^{G_p}_{P_p} \pi \to V_p$$

with $P$ (upper-triangular) parabolic with $H$ as Levi
and $\pi$ irreducible representation of $H_p$

Example (case $GL_2$): principal series have two, special
representations one, supercuspidal zero stabilizations

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Idea: \( \text{Ind}^{G_p}_{P_p} \pi \) is a sheaf on \( P_p \backslash G_p \)

trivialize it on open subset of open Bruhat cell:

\[
\begin{pmatrix}
g_1 & * \\
0 & g_2
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
1 & u \\
0 & 1
\end{pmatrix}
\]

More precisely: Fix \( \rho \in \pi \) and set

\[
\delta(f) \begin{pmatrix}
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The map delta III - the final page

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