Shalika models and p-adic L-function

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Structure of talk

Setup

F totally real number field

$V = \otimes V_v$ cuspidal automorphic representation of GL_{2n}/F such that

- V is cohomological with respect to V_{al}
- V has a Shalika model

Examples:

- f modular form of weight $k \geq 2$ - $V_{al} = \mathrm{Sym}^{k-2}\,\mathbb{C}^2$
- $\rightsquigarrow$ $\mathrm{Sym}^3 f$ "symmetric cube"

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F_∞ maximal abelian unramified outside p and ∞ extension of F

$\mathcal{G}_p = \mathrm{Gal}(F_\infty, F)$

s critical (half-)integer of V

Aim: p-adically interpolate $L(V \otimes \chi, s)$ for

$$\chi : \mathcal{G}_p \to \mathbb{C}^*$$

More precisely: Construct p-adic measure μ on $\mathcal{G}_p$ such that

$$\int_{\mathcal{G}_p} \chi d\mu = E(\chi) \cdot L(V \otimes \chi, s)$$

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Theorem

Theorem (G.)

Suppose

- *V admits a weak ordinary p-stabilization Θ*
- *$V_p \otimes V_{al}$ is cohomologically integral*

Then such a measure μ exists.

Ash-Ginzburg 1994, Dimitrov-Januszewski-Raghuram 2018

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Modular forms

$$L(f,1) = -2\pi i \int_0^{i\infty} f(z)dz$$

First observation:
cycle from 0 to ∞ is orbit of maximal torus in $GL_2(\mathbb{R})$
Second observation:
If $k = 2$, then

$$f \in H^1_c(\Gamma\backslash\mathbb{H}, \mathbb{C})$$

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$H = GL_n \times GL_n \subset G = GL_{2n}$

$\rightsquigarrow$ inclusion of symmetric spaces $X_H \hookrightarrow X_G$

$\rightsquigarrow$ cycle $[X_H]$ of dimension q on $\Gamma \backslash X_G$

V cohomological with respect to $\mathbb{C}$

$\rightsquigarrow$ Eichler-Shimura map $\rightsquigarrow$ forms $\omega \in H_c^q(\Gamma \backslash X_G, \mathbb{C})$

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V has a Shalika model .i.e. a certain linear form does not vanish
(Friedberg-Jacquet) $\rightsquigarrow L(V \otimes \chi, 1/2)$ is linear combination of period integrals

Ash-Ginzburg: Construct μ by writing down explicit ω
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Open-Minded

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Group cohomology

Remember:

$$H^i(\Gamma\backslash X_G, \mathbb{C}) = H^i(\Gamma, \mathbb{C})$$

What about cohomology with compact support?

Borel-Serre compactification yields

$$H_c^i(\Gamma\backslash X_G, \mathbb{C}) = H^{i-l}(\Gamma, \mathrm{Hom}(D_G, \mathbb{C}))$$

D_G "Steinberg module" - free $\mathbb{Z}$-module of infinite rank
Example: $G = GL_2/F \rightsquigarrow D_G = Div_0(\mathbb{P}^1(F)),\ l = 1$

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Cohomology of p-arithmetic groups

V_p local component of V at p (defined over number field E)
$\rightsquigarrow$ modified Eichler-Shimura map
$\rightsquigarrow$ classes $\omega \in H^{q-l}(\Gamma^{(p)}, \mathrm{Hom}(D_G, \mathrm{Hom}(V_p, \mathbb{C})))$
$\Gamma^{(p)} \subseteq PGL_{2n}(\mathcal{O}[1/p])$ - p-arithmetic group

Evaluation:
$H^{q-l}(\Gamma^{(p)}, \mathrm{Hom}(D_G, \mathrm{Hom}(V_p, \mathbb{C}))) \to H^{q-l}(\Gamma, \mathrm{Hom}(D_G, \mathbb{C}))$
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Rationality

Lemma

The canonical map

$$H^i(\Gamma^{(p)}, Hom(D_G, Hom(V_p, E))) \otimes \mathbb{C}$$
$$\to H^i(\Gamma^{(p)}, Hom(D_G, Hom(V_p, \mathbb{C})))$$

is an isomorphism and the space is finite-dimensional.

Idea of proof: Schneider-Stuhler resolution

$$0 \to \text{c-ind}_{K_0}^{G_p} L_0 \ldots \to \text{c-ind}_{K_r}^{G_p} L_r \to V_p \to 0$$

with L_j finite-dimensional
Shapiro's Lemma $\rightsquigarrow$ reduce to finite-dimensional coefficients

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Rationality

Lemma

The canonical map

$$H^i(\Gamma^{(p)}, Hom(D_G, Hom(V_p, E))) \otimes \mathbb{C}$$
$$\to H^i(\Gamma^{(p)}, Hom(D_G, Hom(V_p, \mathbb{C})))$$

is an isomorphism and the space is finite-dimensional.

Idea of proof: Schneider-Stuhler resolution

$$0 \to \text{c-ind}_{K_0}^{G_p} L_0 \ldots \to \text{c-ind}_{K_r}^{G_p} L_r \to V_p \to 0$$

with L_j finite-dimensional
Shapiro's Lemma $\rightsquigarrow$ reduce to finite-dimensional coefficients

Integrality

What about integrality?

Want integral model $\mathcal{L}$ of V_p (over $\mathbb{Z}$ or $\mathbb{Z}_p$)
with resolution as before - "cohomologically integral"
$\rightsquigarrow$ get integral structure on cohomology

Known in many cases:
Vignéras (GL_2), Große-Klönne (spherical), Ollivier (ordinary)

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The map delta I

Idea: construct a map

$$\delta : \text{function space} \to V_p$$

Pullback to H yields

$$\delta^\vee : H^{q-l}(\Gamma^{(p)}, \mathrm{Hom}(D_G, \mathrm{Hom}(V_p, E)))$$
$$\to H^{q-l}(\Gamma_H^{(p)}, \mathrm{Hom}(D_H, \text{distributions}))$$

Pushforward along determinant gives μ

"weakly ordinary" $\rightsquigarrow$ δ respects integral models
"cohomologically integral" + "weakly ordinary" $\Rightarrow$ μ measure

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The map delta II

Assumption: V_p has a stabilization i.e. a non-trivial map

$$\Theta\colon \operatorname{Ind}_{P_p}^{G_p} \pi \to V_p$$

with P (upper-triangular) parabolic with H as Levi and π irreducible representation of H_p

Example (case GL_2): principal series have two, special representations one, supercuspidal zero stabilizations

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The map delta III - the final page

Idea: $\mathrm{Ind}_{P_p}^{G_p}\pi$ is a sheaf on $P_p\backslash G_p$

trivialize it on open subset of open Bruhat cell:

$$\begin{pmatrix} g_1 & * \\ 0 & g_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

More precisely: Fix $\rho \in \pi$ and set

$$\delta(f)\left(\begin{pmatrix} g_1 & * \\ 0 & g_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) = f(u)\cdot\pi\left(g_1, g_2 u\right)\rho$$

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