On exceptional zeros for GL(2)

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June 10th, 2020

Structure of talk

Mazur–Tate–Teitelbaum conjecture

$f = \sum_n a_n q^n \in S_{2k}(N)$ newform, $N = pM, p \nmid M$
$\rightsquigarrow L_p(f, s)$ p-adic L-function

$$L_p(f, k) \approx (1 - p^{k-1}/a_p) \cdot L(f, k)$$

$a_p = p^{k-1} \rightsquigarrow$ exceptional zero
Conjecture (Mazur–Tate–Teitelbaum):

$$\frac{d}{ds} L_p(f, s)|_{s=k} \approx \mathcal{L}(f) \cdot L(f, k)$$

$\mathcal{L}(f)$ is a local invariant, i.e. $\mathcal{L}(f)$ only depends on the restriction of p-adic Galois representation attached to f to a decomposition group at p

Tate periods

$2k = 2$, $a_n \in \mathbb{Q}$ for all n
$f \longleftrightarrow$ elliptic curve $E_f/\mathbb{Q}$
$N = pM, p \nmid M$, $a_p = 1$
$\Rightarrow E_f$ has split multiplicative reduction at p

Theorem (Tate)

a. $q \in \mathbb{Q}_p$, $0 < |q|_p < 1$

$$\rightsquigarrow E_q/\mathbb{Q}_p \text{ such that } E_q(\mathbb{C}_p) \cong \mathbb{C}_p^*/q^{\mathbb{Z}}$$

b. *If $E/\mathbb{Q}_p$ has split multiplicative reduction*

$$\rightsquigarrow E \cong E_q \text{ for a unique } q \in \mathbb{Q}_p,\ 0 < |q|_p < 1$$

$\mathcal{L}$-invariants

$$\mathcal{L}(E_q) = \frac{\log_p(q)}{\mathrm{ord}_p(q)} \rightsquigarrow \text{isogeny invariant}$$
$$\mathcal{L}(f) = \mathcal{L}(E_{f,\mathbb{Q}_p})$$

MTT conjecture proven by Greenberg–Stevens
What about higher weights?

- Definition of $\mathcal{L}$-invariants by Fontaine–Mazur, Teitelbaum, Coleman, Darmon, Orton, Breuil
- Proof of MTT by Kato–Kurihara–Tsuji, Stevens, Darmon, Orton, Emerton
- Equality of $\mathcal{L}$-invariants by Coleman–Iovita, Iovita-Spieß, Breuil, Bertolini-Darmon-Iovita

GL_2 over other number fields

F number field, π cohomological automorphic representation of GL_2/F

small slope condition $\rightsquigarrow L_p(f, \underline{s})$ Barrera-Williams

new phenomenons:

- function in many variables if F is not totally real
  $\rightsquigarrow$ need to compute partial derivatives in several directions
- $L_p(f, k) \approx \prod_{\mathfrak{p}|p} e_{\mathfrak{p}}(k) \cdot L(f, k)$
  multiple $e_{\mathfrak{p}}(k) = 0 \rightsquigarrow$ compute higher partial derivatives

Known cases:

- F totally real, parallel weight 2, first order: Mok
- F totally real, parallel weight 2: Spieß
- F arbitrary, parallel weight 2: Deppe, Bergunde
- F totally real: Barrera–Dimitrov–Jorza
- F imaginary quadratic, first order: Barrera–Williams

$\mathbb{Z}_p$-extensions

F number field, F_∞/F extension with $\mathcal{G} = \mathrm{Gal}(F_\infty/F) \cong \mathbb{Z}_p$
Example: $\mathrm{Gal}(F(\mu_{p^\infty})/F) \cong \mathbb{Z}_p \times$ finite
F_∞ is unramified outside p
put $F_p = F \otimes \mathbb{Q}_p \cong \prod_{\mathfrak{p}|p} F_\mathfrak{p}$, $\mathcal{O}_{F,S_p} = \mathcal{O}_F[1/p]$
$\rightsquigarrow$ class field theory:

$$\mathrm{rec}\colon F_p^*/\mathcal{O}_{F,S_p}^* \twoheadrightarrow \mathcal{G}$$

Fix isomorphism $\ell\colon \mathrm{Gal}(F_\infty/F) \xrightarrow{\cong} p\mathbb{Z}_p$ (resp. $4\mathbb{Z}_2$ if $p=2$)

$$\begin{aligned}\chi_\ell\colon F_p^*/\mathcal{O}_{F,S_p}^* &\longrightarrow C^{\mathrm{an}}(\mathbb{Z}_p,\mathbb{Z}_p)^* \\ \gamma &\longmapsto [s \mapsto \exp_p(s\cdot\ell(\mathrm{rec}(\gamma)))]\end{aligned}$$

Homology classes

R a $\mathbb{Q}_p$-algebra, $\chi\colon F_p^*/\mathcal{O}_{F,S_p}^* \to R^*$ locally analytic character
$\rightsquigarrow \chi \in H^0(\mathcal{O}_{F,S_p}^*, C^{\mathrm{an}}(F_p^*, R))$
$E \subseteq \mathcal{O}_F^*$ torsion-free, finite index
Dirichlet's unit theorem $\Rightarrow E \cong \mathbb{Z}^r$, $r = r_{\mathbb{R}} + r_{\mathbb{C}} - 1$

$$\Rightarrow H_r(E, \mathbb{Z}) \cong \mathbb{Z}$$

Shapiro $\rightsquigarrow$ fundamental class $\vartheta \in H_r(\mathcal{O}_{F,S_p}^*, C_c(F_p^*/U_p, \mathbb{Z}))$

$$c_\chi = \chi \cap \vartheta \in H_r(\mathcal{O}_{F,S_p}^*, C_c^{\mathrm{an}}(F_p^*, R))$$

Given $\kappa \in H^r(\mathcal{O}_{F,S_p}^*, \mathrm{Dist}(F_p^*, E))$, $E/\mathbb{Q}_p$ finite

$$\rightsquigarrow L_p(\kappa, \ell, s) = \kappa \cap \chi_\ell \in C^{\mathrm{an}}(\mathbb{Z}_p, E)$$

Cohomology of locally symmetric spaces

π cohomological cuspidal automorphic representation of $\mathrm{PGL}_{2,F}$

$$\Rightarrow H_c^*(\Gamma_0(\mathfrak{n})\backslash X, V_{\underline{k}}^\vee)_{\mathfrak{m}_\pi} \neq 0$$

$\Gamma_0(\mathfrak{n}) \subseteq \mathrm{PGL}_2(\mathcal{O}_F)$ congruence subgroup
$X = \mathbb{H}_2^{r_\mathbb{R}} \times \mathbb{H}_3^{r_\mathbb{C}}$ symmetric space
V_k algebraic representation of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{PGL}_{2,F}$
m_π ideal of Hecke algebra given by π
Moreover:

$$\dim H_c^{r+1}(\Gamma_0(\mathfrak{n})\backslash X, V_{\underline{k}}^\vee)_{\mathfrak{m}_\pi}^\epsilon = 1$$

Put $\Delta_0 = \ker(\mathbb{Z}[\mathbb{P}^1(F)] \to \mathbb{Z}, \sum_x n_x x \mapsto \sum_x n_x)$

$$\rightsquigarrow H^i(\Gamma_0(\mathfrak{n}), \mathrm{Hom}(\Delta_0, M)) \xrightarrow{\cong} H_c^{i+1}(\Gamma_0(\mathfrak{n})\backslash X, M)$$

$$\Rightarrow H^r(\Gamma_0(\mathfrak{n}), \mathrm{Hom}(\Delta_0, V_{\underline{k}}^\vee))_{\mathfrak{m}_\pi}^\epsilon = 1$$

Overconvergent cohomology

$I_p = \prod_{\mathfrak{p}|p} I_{\mathfrak{p}} \subseteq \mathrm{PGL}_2(\mathcal{O}_{F_p})$ Iwahori subgroup

$$I_{\mathfrak{p}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathcal{O}_{F_{\mathfrak{p}}}) \;\middle|\; \mathfrak{p} \text{ divides } c \right\}$$

$$\mathcal{A}_{\underline{k}} = \left\{ f\colon I_p \to E \text{ locally analytic} \;\middle|\; f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = (d/a)^{\underline{k}} f(g) \right\}$$

Have embedding $V_{\underline{k}} \hookrightarrow \mathcal{A}_{\underline{k}}$

$$\rightsquigarrow H^i(\Gamma_0(\mathfrak{n}), \mathrm{Hom}(\Delta_0, \mathcal{A}_{\underline{k}}^\vee))^{\epsilon}_{\mathfrak{m}_\pi} \to H^i(\Gamma_0(\mathfrak{n}), \mathrm{Hom}(\Delta_0, V_{\underline{k}}^\vee))^{\epsilon}_{\mathfrak{m}_\pi}$$

Ash-Stevens: This is an isomorphism for small slopes. In particular:

$$H^r(\Gamma_0(\mathfrak{n}), \mathrm{Hom}(\Delta_0, \mathcal{A}_{\underline{k}}^\vee))^{\epsilon}_{\mathfrak{m}_\pi} = 1$$

Cohomology of p-arithmetic groups

$\Gamma_0(\mathfrak{n})^p \subseteq \mathrm{PGL}_2(\mathcal{O}_{F,S_p})$ p-arithmetic congruence subgroup

$$\rightsquigarrow H^r(\Gamma_0(\mathfrak{n})^p, \mathrm{Hom}(\Delta_0, I(\chi)^\vee))^{\epsilon}_{\mathfrak{m}_\pi} \xrightarrow{\cong} H^r(\Gamma_0(\mathfrak{n}), \mathrm{Hom}(\Delta_0, \mathcal{A}^\vee_{\underline{k}}))^{\epsilon}_{\mathfrak{m}_\pi}$$

$I(\chi)$ locally analytic principal series:

$$I(\chi) = \left\{ f\colon G(F_p) \to E \text{ locally analytic} \;\middle|\; f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi(d/a) f(g) \right\}$$

$\chi\colon F_p^* \to E^*$, $\chi(x) = x^{\underline{k}}$ if x integral, $\chi(\varpi) = U_{\mathfrak{p}}$-eigenvalue
Proof via Koszul resolution by Kohlhaase–Schraen

The map δ

Embed F_p^* in $\mathrm{PGL}_2(F_p)$ via $x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$
Want F_p^*-equivariant map

$$\delta\colon\, C_c^{\mathrm{an}}(F_p^*, E) \to I(\chi)$$

$$\delta^*\colon\, H^r(\Gamma_0(\mathfrak{n})^p, \mathrm{Hom}(\Delta_0, I(\chi)^\vee))^\epsilon_{\mathfrak{m}_\pi} \to H^r(\mathcal{O}^*_{F,S_p}, \mathrm{Dist}(F_p^*, E))$$

$\rightsquigarrow \kappa_\pi \rightsquigarrow L_p(\pi, \ell, s) = L_p(\kappa_\pi, \ell, s)$

δ is given by trivialization of $I(\chi)$ on open subset of $\mathbb{P}^1$:
$\delta(f)(g) = f(u)\chi(u)\chi(d/a)$ for $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$

The map δ again

$\chi = \prod_{\mathfrak{p}|p} \chi_{\mathfrak{p}}$ with $\chi_{\mathfrak{p}}\colon F_{\mathfrak{p}}^* \to E^*$
Suppose $\chi_{\mathfrak{p}}$ can be extended to analytic function on $F_{\mathfrak{p}}$ for $\mathfrak{p} \in S_{\mathrm{sp}} \subseteq S_p$

$$\delta\colon C_c^{\mathrm{an}}\Big(\prod_{\mathfrak{p}\in S_{\mathrm{sp}}} F_{\mathfrak{p}} \times \prod_{\mathfrak{p}\notin S_{\mathrm{sp}}} F_{\mathfrak{p}}^*, E\Big) \to I(\chi)$$

$\Rightarrow \kappa_\pi \in H^r(\mathcal{O}_{F,S_p}^*, \mathrm{Dist}(\prod_{\mathfrak{p}\in S_{\mathrm{sp}}} F_{\mathfrak{p}} \times \prod_{\mathfrak{p}\notin S_{\mathrm{sp}}} F_{\mathfrak{p}}^*, E))$
Spieß, Dasgupta-Spieß $\rightsquigarrow$ $\mathrm{ord}_{s=0} L_p(\pi, \ell, s) \geq |S_{\mathrm{sp}}|$
+ cohomological formula for $|S_{\mathrm{sp}}|$-th derivative

Lemma

$\chi_{\mathfrak{p}}$ can be extended to analytic function on $F_{\mathfrak{p}}$ if and only if $\chi_{\mathfrak{p}}(x) = x^{k_{\mathfrak{p}}}$ for all x

The Theorem

Theorem

Let $n = |S_{\mathrm{sp}}|$.

$$\frac{d^n}{ds^n} L(\pi, \ell, s)|_{s=0} \approx \prod_{\mathfrak{p} \in S_{\mathrm{sp}}} \mathcal{L}_{\ell}^{\mathrm{aut}}(\pi, \mathfrak{p}) \cdot \prod_{\mathfrak{p} \notin S_{\mathrm{sp}}} e_{\mathfrak{p}}(1/2) \cdot L(\pi, 1/2)$$

Automorphic $\mathcal{L}$-invariants I

Let $\mathfrak{p} \in S_{\mathrm{sp}}$. Want to construct:

$$\mathrm{Hom}(F_{\mathfrak{p}}^{*}, E) \to E,\ \lambda \mapsto \mathcal{L}_{\lambda}(\pi, \mathfrak{p})$$

such that $\mathcal{L}_{\mathrm{ord}_{\mathfrak{p}}}(\pi, \mathfrak{p}) = 1$
$\rightsquigarrow \mathcal{L}_{\ell}^{\mathrm{aut}}(\pi, \mathfrak{p}) = \mathcal{L}_{\ell \circ \mathrm{rec}_{\mathfrak{p}}}^{\mathrm{aut}}(\pi, \mathfrak{p})$
Two steps:
(i): $\mathfrak{p} \in S_{\mathrm{sp}} \Rightarrow I(\chi_{\mathfrak{p}}) \twoheadrightarrow \mathrm{St}(\underline{k}_{\mathfrak{p}})^{\mathrm{an}}$

$$\rightsquigarrow \dim H^{r}(\Gamma_0(\mathfrak{n})^{\mathfrak{p}}, \mathrm{Hom}(\Delta_0, (\mathrm{St}(\underline{k}_{\mathfrak{p}})^{\mathrm{an}})^{\vee}))_{\mathfrak{m}_{\pi}}^{\epsilon} = 1$$

(ii): Breuil $\rightsquigarrow \mathrm{Hom}(F_{\mathfrak{p}}^{*}, E) \to \mathrm{Ext}^{1}_{\mathrm{PGL}_2(F_{\mathfrak{p}})}(V_{\underline{k}_{\mathfrak{p}}}, \mathrm{St}(\underline{k}_{\mathfrak{p}})^{\mathrm{an}})$

$$H^{r}(\Gamma_0(\mathfrak{n})^{\mathfrak{p}}, \mathrm{Hom}(\Delta_0, (\mathrm{St}(\underline{k}_{\mathfrak{p}})^{\mathrm{an}})^{\vee}))_{\mathfrak{m}_{\pi}}^{\epsilon} \xrightarrow{\cup \lambda} H^{r+1}(\Gamma_0(\mathfrak{n})^{\mathfrak{p}}, \mathrm{Hom}(\Delta_0, V_{\underline{k}_{\mathfrak{p}}}^{\vee}))_{\mathfrak{m}_{\pi}}^{\epsilon}$$

Automorphic $\mathcal{L}$-invariants II

Lemma

The cup product with the $\mathfrak{p}$-adic valuation $\mathrm{ord}_{\mathfrak{p}}$ is an isomorphism.

Definition

$$\mathcal{L}_{\lambda}^{\mathrm{aut}}(\pi, \mathfrak{p})^{\epsilon} = (\cup\, \mathrm{ord}_{\mathfrak{p}})^{-1} \circ (\cup\, \lambda)$$