1. Projective and quasi-projective varieties

References: [1, Chap. I, §4, §5], [2, Chapter 4, Chapter 6], [4, Lecture 1, Lecture 2]

(1) Basic notions [1, §4.1] (1 lecture)
- definition of $\mathbb{P}^n_k := \{k^\times \backslash (k^{n+1} \setminus \{0\})\} = \{\text{lines in } k^{n+1}\}$
- closed subsets of projective space, homogeneous polynomials, homogeneous ideals. The operations: $I_h(S) \subset k[X_0, \ldots, X_n] :=$ homogeneous ideal of a subset $S \subset \mathbb{P}^n_k$, $V_h(I) \subset \mathbb{P}^n_k :=$ zero-set of a homogeneous ideal $I \subset k[X_0, \ldots, X_n]$. [1, Lemma, pg. 44]: $V_h(I) = 0 \Leftrightarrow I \supset (X_0, \ldots, X_n)^m$ for some $m \geq 0$.
- the Zariski topology on $\mathbb{P}^n_k$, principal affine open subsets $U_i = \{(x_0 : \ldots : x_n) \mid x_i n \neq 0\} \cong k^n$, dehomogenized ideal of $X \cap U_i$, homogeneous ideal of $\bar{X}$ for $X \subset U_i$ closed, quasi-projective varieties with the Zariski topology.

(2) regular functions [1, §4.2] (1 lecture)
- local definition as ratio of homogeneous polynomials, global rings of functions
- for $X \subset U_i \cong k^n$, identification of $k[X]$ with affine coordinate ring of $X \subset k^n$
- regular maps (morphisms) of quasi-projective varieties and the special case of $f\colon X \to \mathbb{P}^m$
- affine varieties, local properties: [1, Lemma 1, Lemma 2, pg. 49]
- principal open subsets

(3) products [1, §5.1] (1 lecture)
- products of open subsets of affine varietes, products of projective spaces
- closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ [1, §5.1, Theorem 1]

2. Grassmannian varieties

References: [1, pg. 42, Example 1], [3, Chap. 5, §2], [5, Part II, §5.1-5.3], [4, Lecture 6]

(1) definitions and first properties (1 lecture)
- definition of $\mathrm{Gr}(r, n+r)$ as $\{r \text{ planes in } k^{n+r}\} = \{\mathbb{P}^{r-1}s \text{ in } \mathbb{P}^{n+r-1}\}$, hence a generalization of $\mathbb{P}^n = \mathrm{Gr}(1, n+1)$.
- definition as quotient $\mathrm{GL}_r(k) \backslash \mathcal{U}_{r,n+r}$, where $\mathcal{U}_{r,n+r} = \{M \in M_{r \times n+r} \mid \mathrm{rank}(M) = r\}$
- covering by principal open subsets $U_J \cong k^{rn}$, $J = (j_1, \ldots, j_r)$, $1 \leq j_1 < \ldots < j_r \leq n+r$, $U_I =$ image of $\mathcal{U}^I_{r,n+r} = \{M = (m_{ij}) \in M_{r \times n+r} \mid \det(m_{ij_i}) \neq 0\}$

(2) Plücker embedding and Plücker relations (1 lecture)
- Plücker embedding: show that $\mathrm{Gr}(r, n+r)$ has the structure of a projective variety in $\mathbb{P}^{N_{r,n}}$, $N_{r,n} = \binom{n+r}{r} - 1$.
- Plücker relations
- example: $\mathrm{Gr}(2,4)$ as a quadric in $\mathbb{P}^5$

(3) open Schubert cells and Schubert varieties (2 lectures)
- definition of Schubert cells and Schubert varieties

- stratification of $\mathrm{Gr}(r, n+r)$ into affine spaces. Do the example of $\mathbb{P}^n$ and $\mathrm{Gr}(2,4)$.
- parametrization through partitions, dual partition, dual Schubert varieties and Poincaré duality.

3. Vector bundles on Grassmannians

References: [3, Chap. 5, §2], [4, Lecture 16, pg. 200-202]

(1) vector bundles (1 lecture)
   - definition of vector bundles via open cover and local trivialization with GL_r-transition maps
   - short exact sequences of vector bundles, determinant bundles

(2) vector bundles on Grassmannians (2 lectures)
   - representation of $\mathrm{Gr}(r, n+r)$ as $\mathrm{GL}_{n+r}/P_{r,n}$,
   - constructing a vector bundle on $\mathrm{Gr}(r, n+r)$ from a representation of $P_{r,n}$
   - examples: the tautological sequence $0 \to R \to O^{n+r}_{\mathrm{Gr}(r,n+r)} \to Q \to 0$, the line bundle $O(1) := \det Q$, the tangent bundle $T_{\mathrm{Gr}(r,n+r)} \cong Q \otimes R^\vee$

4. Surprise topic

1 lecture, talk could include one of the following topics: singular cohomology of Grassmannians [3, pg. 106, Thm.P], universal property of $\mathrm{Gr}(r, n+r)$ [3, Thm. N, pg. 100], Schubert varieties as Chern classes [3], applications to geometric problems such as lines on degree $2d-1$ hypersurfaces in $\mathbb{P}^{d+1}$, partial flag varieties (varieties of incidence planes), planes meeting a variety $X \subset \mathbb{P}^n$

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[3] P. Griffiths, **Topics in Algebraic and Analytic Geometry** Written and revised by John Adams. Reprint of the 1974 edition. Princeton Legacy Library. Princeton University Press, Princeton, NJ, [2015].

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