Big principal series and L-invariants

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Structure of talk

Elliptic curves

k field, $char(k) \neq 2, 3$

E/k elliptic curve, $E = V(Y^2 - X^3 - aX - b)$ with $a, b \in k$ such that $4a^3 + 27b^2 \neq 0$

Alternatively: E/k connected, smooth, projective curve of genus 1 with k-rational point

Alternatively: E/k one-dimensional, connected, projective, algebraic group

Complex uniformization

$k = \mathbb{C}$
$\rightsquigarrow E(\mathbb{C})$ is a complex one-dimensional torus, i.e.:

$$E(\mathbb{C}) \cong \mathbb{C}/\Lambda,\ \Lambda \subseteq \mathbb{C} \text{ lattice}$$

Wlog $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, $\mathfrak{Im}(\tau) > 0$

$$\mathbb{C}/\mathbb{Z} \xrightarrow{\exp(2\pi i\ \cdot)} \mathbb{C}^*$$

yields

$$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}},\ 0 < |q| < 1$$

P-adic uniformization

What about $k = \mathbb{Q}_p$?
$\Lambda \subseteq \mathbb{Q}_p$ discrete $\rightsquigarrow \Lambda = \{0\}$
But: $q \in \mathbb{Q}_p$, $0 < |q|_p < 1 \rightsquigarrow q^{\mathbb{Z}} \subseteq \mathbb{Q}_p^*$ discrete, cocompact

Theorem (Tate 1959)

a. $q \in \mathbb{Q}_p$, $0 < |q|_p < 1$

$$\rightsquigarrow E_q/\mathbb{Q}_p \text{ such that } E_q(\mathbb{Q}_p) \cong \mathbb{Q}_p^*/q^{\mathbb{Z}}$$

b. *If $E/\mathbb{Q}_p$ has split multiplicative reduction*

$$\rightsquigarrow E \cong E_q,\ q \in \mathbb{Q}_p,\ 0 < |q|_p < 1$$

Modularity

$k = \mathbb{Q}$

(Wiles et al.) $\rightsquigarrow \exists\, f \in S_2(\Gamma_0(N))$ such that

$$L(E, s) = L(f, s)$$

f is a differential form on a Riemann surface
$\rightsquigarrow$ purely analytic object

Suppose E has split multiplicative reduction at prime p
(Tate) $\rightsquigarrow$ $q_E \in \mathbb{Q}_p^*$

Question: Can we recover q_E from f?

Arithmetic L-Invariants

Answer: No!

$E \sim E'$ isogenous $\rightsquigarrow L(E,s) = L(E',s)$
but: $q_E^n = q_{E'}^m$ for $m, n \in \mathbb{Z}_{>0}$

$$\rightsquigarrow \mathcal{L}_p(E) = \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)} \text{ isogeny invariant}$$

Question: Can we recover $\mathcal{L}_p(E)$ from f?

A zoo of L-Invariants

Answer: Yes!

Teitelbaum: $\mathcal{L}^T(f)$ using Cerednik-Drinfeld uniformization
Darmon: $\mathcal{L}^D(f)$ using cohomology of $\mathrm{SL}_2(\mathbb{Z}[1/p])$
Breuil: $\mathcal{L}^B(f)$ using completed cohomology
Greenberg-Stevens: $a_p' =$ derivative of U_p-eigenvalue of Hida family through f

Theorem (Bertolini-Darmon-Iovita 2010)

$$\mathcal{L}^D(f) = -2a_p'$$

Aim: give new proof that generalizes to higher rank groups (joint work with Giovanni Rosso)

1. Motivation: p-adic periods of elliptic curves
2. Darmon's L-invariant
3. Big principal series and p-adic families

Step 1: Cohomology of arithmetic groups

$E \longleftrightarrow f$ weight 2 modular form of level N

$$\Gamma_0(N) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}$$

$\rightsquigarrow f$ (holomorphic) 1-form on $\Gamma_0(N)\backslash\mathbb{H}$

$$f \in \mathrm{H}^1(\Gamma_0(N)\backslash\mathbb{H}, \mathbb{C}) \cong \mathrm{H}^1(\Gamma_0(N), \mathbb{C})$$

Multiplicity one theorem:

$$\dim \mathrm{H}^1(\Gamma_0(N), \mathbb{C})^+[f] = 1$$
$$\dim \mathrm{H}^1(\Gamma_0(N), \mathbb{Q}_p)^+[f] = 1$$

Step 2: Cohomology of p-arithmetic groups

E split multiplicative reduction at p

$$\Gamma_0^p(N) = \left\{\gamma \in \mathrm{GL}_2(\mathbb{Z}[1/p]) \mid \gamma \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N,\ \det(\gamma) > 0\right\}$$

$$\mathrm{St}_p(R) = \left\{\phi\colon \mathbb{P}^1(\mathbb{Q}_p) \longrightarrow R \mid \phi \text{ locally constant}\right\} / \{\text{constant}\}$$

There is a natural isomorphism of 1-dim vector spaces:

$$\mathrm{H}^1(\Gamma_0^p(N), \mathrm{Hom}_{\mathbb{Q}_p}(\mathrm{St}_p(\mathbb{Q}_p), \mathbb{Q}_p))^+[f] \xrightarrow{\cong} \mathrm{H}^1(\Gamma_0(N), \mathbb{Q}_p)^+[f]$$

Flat base change holds:

$$\mathrm{H}^1(\Gamma_0^p(N), \mathrm{Hom}_{\mathbb{Z}_p}(\mathrm{St}_p(\mathbb{Z}_p), \mathbb{Z}_p)) \otimes \mathbb{Q}_p$$
$$\xrightarrow{\cong} \mathrm{H}^1(\Gamma_0^p(N), \mathrm{Hom}_{\mathbb{Q}_p}(\mathrm{St}_p(\mathbb{Q}_p), \mathbb{Q}_p))$$

Step 3: P-adic integration

Remember $\log_p(q_E) = \mathcal{L}_p(E) \cdot \mathrm{ord}_p(q_E)$

Breuil:

$\rightsquigarrow \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^*, \mathbb{Q}_p) \longrightarrow \mathrm{Ext}^1(\mathbb{Q}_p, \mathrm{St}_p^{\mathrm{cont}}(\mathbb{Q}_p)),\ \lambda \longmapsto \mathcal{E}(\lambda)$

$$\rightsquigarrow c_\lambda :\ \mathrm{H}^1(\Gamma_0^p(N), \mathrm{St}_p(\mathbb{Q}_p)^\vee)^+[f] \xrightarrow{\cup \mathcal{E}(\lambda)} \mathrm{H}^2(\Gamma_0^p(N), \mathbb{Q}_p)^+[f]$$

Fact: c_{ord_p} is an isomorphism of 1-dim vector spaces

Definition (Darmon 01)

$$c_{\log_p} = \mathcal{L}_p^D(E) \cdot c_{\mathrm{ord}_p}$$

Breuil's extensions

$$\mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^*, \mathbb{Q}_p) \longrightarrow \mathrm{Ext}^1(\mathbb{Q}_p, \mathrm{St}_p^{\mathrm{cont}}(\mathbb{Q}_p)),\ \lambda \longmapsto \mathcal{E}(\lambda)$$

$$\tau_\lambda \colon \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longmapsto \begin{pmatrix} 1 & \lambda(a/d) \\ 0 & 1 \end{pmatrix}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\mathrm{Ind}_B^G \mathbb{Q}_p)^{\mathrm{cont}} & \longrightarrow & (\mathrm{Ind}_B^G \tau_\lambda)^{\mathrm{cont}} & \longrightarrow & (\mathrm{Ind}_B^G \mathbb{Q}_p)^{\mathrm{cont}} & \longrightarrow & 0 \\ & & \downarrow\!\!\downarrow & & & & \uparrow & & \\ & & \mathrm{St}_p(\mathbb{Q}_p)^{\mathrm{cont}} & & & & \mathbb{Q}_p & & \end{array}$$

A reformulation

Remember $c_{\log_p} = \mathcal{L}_p^D(E) \cdot c_{\mathrm{ord}_p}$

$\dim \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}_p^*, \mathbb{Q}_p) = 2$ with basis $\{\mathrm{ord}_p, \log_p\}$
$\rightsquigarrow \langle \log_p - \mathcal{L}_p^D(E) \cdot \mathrm{ord}_p \rangle = \ker(\lambda \mapsto c_\lambda)$

Question: How to produce a non-zero class in $\ker(\lambda \mapsto c_\lambda)$?

$$c_\lambda = 0 \iff$$
$$\mathrm{H}^1(\Gamma_0^p(N), \mathcal{E}(\lambda)^\vee)_f^+ \xrightarrow{\neq 0} \mathrm{H}^1(\Gamma_0^p(N), \mathrm{St}_p(\mathbb{Q}_p)^\vee)^+[f]$$

Big principal series

$$\tau_\lambda \colon B \longrightarrow \mathrm{GL}_2(\mathbb{Q}_p),\ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \longmapsto \begin{pmatrix} 1 & \lambda(a/d) \\ 0 & 1 \end{pmatrix}$$

$\tau_\lambda \longleftrightarrow \chi_\lambda \colon \mathbb{Q}_p^* \longrightarrow (\mathbb{Q}_p[\varepsilon]/\varepsilon^2)^*$ such that $\chi_\lambda \equiv 1 \bmod \varepsilon$

Idea: replace $\mathbb{Q}_p[\varepsilon]/\varepsilon^2$ by Tate algebra $\mathcal{O} = \mathbb{Q}_p\langle T \rangle$

Want: $\chi_\mathcal{O} \colon \mathbb{Q}_p^* \longrightarrow \mathcal{O}^*$ such that

- $\chi_\mathcal{O} \equiv 1 \bmod (T)$ and
- $\mathrm{H}^1(\Gamma_0^p(N), \mathrm{Hom}_\mathcal{O}((\mathrm{Ind}_B^G \chi_\mathcal{O})^{\mathrm{an}}, \mathcal{O}))_f^+ \neq 0$

$\rightsquigarrow$ reduce $\chi_\mathcal{O}$ modulo (T^2) to get λ with $c_\lambda = 0$

Kohlhaase-Schraen resolution

Consider $\mathcal{A} = (\mathrm{Ind}_{B\cap I}^{I}\,\chi_{\mathcal{O}})^{\mathrm{an}}$, where
$I = \{\gamma \in \mathrm{GL}_2(\mathbb{Z}_p) \mid \gamma \text{ upper triangular modulo } p\}$

Theorem (Kohlhaase-Schraen 2012)

$$0 \longrightarrow \text{c-ind}_I^G\,\mathcal{A} \xrightarrow{U_p - \chi_{\mathcal{O}}(p)} \text{c-ind}_I^G\,\mathcal{A} \longrightarrow (\mathrm{Ind}_B^G\,\chi_{\mathcal{O}})^{\mathrm{an}} \longrightarrow 0$$

Remark: Theorem holds for more general groups
Multiple U_p-operators $\rightsquigarrow$ Koszul resolution

Overconvergent cohomology in families

Choose $\chi_{\mathcal{O}}|_{\mathbb{Z}_p^*}$ as universal character on weight space , i.e.:

$$\chi_{\mathcal{O}}(x)(T) := \exp_p(T \cdot \log_p(x))$$

Overconvergent families à la Ash-Stevens
$\rightsquigarrow \mathrm{H}^1(\Gamma_0(N), \mathcal{A}^\vee)_f^+$ free $\mathcal{O}$-module of rank 1
U_p operates via eigenvalue $a_p \in \mathcal{O}^*$

$$\chi_{\mathcal{O}}(x)(T) := a_p(T)^{\mathrm{ord}_p(x)} \cdot \exp_p(T \cdot \log_p(x))$$

$a_p(0) = a_p(f) = 1 \Rightarrow \chi_{\mathcal{O}} \equiv 1 \bmod (T)$
KS resolution $\Rightarrow \mathrm{H}^1(\Gamma_0^p(N), \mathrm{Hom}_{\mathcal{O}}((\mathrm{Ind}_B^G \chi_{\mathcal{O}})^{\mathrm{an}}, \mathcal{O}))_f^+ \neq 0$

$$\chi_{\mathcal{O}}(x) \mod T^2 = 1 + T\left(\frac{d}{dt} a_p\Big|_{t=0} \cdot \mathrm{ord}_p(x) + \log_p(x)\right)$$