On quaternionic rigid meromorphic cocycles

Lennart Gehrmann

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Motivation

Aim: generalize rigid meromorphic cocycles à la Darmon-Vonk

Q: What is a rigid meromorphic cocycle?

$$\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p) \quad p\text{-adic upper half plane}$$
$$\mathcal{M} = \text{ rigid meromorphic functions on } \mathcal{H}_p$$

A: $J \in H^1 \;\; (\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathcal{M}^\times)$ rigid meromorphic cocycle

Theorem (Darmon-Vonk):

$$\operatorname{supp} \operatorname{Div}(J) \subseteq \text{finite union of RM-orbits}$$

More precise aim: generalize result to $\mathfrak{p}$-arithmetic subgroups of inner forms of SL_2 over arbitrary number fields

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Study cohomology of $\mathrm{Div}^{\dagger}\mathcal{H}_p$ instead of $\mathcal{M}^{\times}$

$$\mathrm{Div}^{\dagger}\mathcal{H}_p = \text{ locally finite divisors}$$

Two ingredients:

1. description of $\mathrm{Div}^{\dagger}\mathcal{H}_p$ in terms of coinduction **and** induction
   - uses reduction map to Bruhat-Tits tree
2. Cohomological properties of arithmetic groups , i.e.:
   - Bieri-Eckmann duality
   - cohomology commutes with direct limits

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Basic idea II - coordinate-free formulation

- F number field , B/F quaternion algebra that is split at $\mathfrak{p}$
- G = group of norm one elements of B

$$\rightsquigarrow G(F_{\mathfrak{p}}) \cong \mathrm{SL}_2(F_{\mathfrak{p}}) \quad \text{non-canonically!}$$

$$\rightsquigarrow G(F_{\mathfrak{p}}) \curvearrowright \mathcal{H}_{\mathfrak{p}} \quad \text{non-canonically!}$$

Solution: replace $\mathbb{P}^1$ by Brauer-Severi variety of B

$$\mathbb{P}_B(E) = \{I \triangleleft B \otimes_F E \text{ left ideal} \mid \dim_E I = 2\}$$

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but $G(F_\mathfrak{p}) \curvearrowright \mathcal{H}_{B_\mathfrak{p}}$ canonically!

Brauer-Severi variety

$$\mathbb{P}_B(E) = \{I \lhd B \otimes_F E \text{ left ideal} \mid \dim_E I = 2\}$$

$$\mathbb{P}_B(E) \neq 0 \iff E \text{ splits } B$$

in that case: $\mathbb{P}_B(E) \cong \mathbb{P}^1(E)$ non-canonically!

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- $\Gamma^{\mathfrak{p}} \subseteq G(F)$ $\mathfrak{p}$-arithmetic subgroup
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Lemma

The canonical map

$$\bigoplus_{\Gamma^{\mathfrak{p}} x \in \Gamma^{\mathfrak{p}} \backslash X} \mathrm{H}^i(\Gamma^{\mathfrak{p}}, \mathrm{Div}^{\dagger} \Gamma^{\mathfrak{p}} x) \longrightarrow \mathrm{H}^i(\Gamma^{\mathfrak{p}}, \mathrm{Div}^{\dagger} X)$$

is an isomorphism.

A bunch of reduction steps

- $o_1, \dots, o_h$ the $\Gamma^{\mathfrak{p}}$-orbits of edges/vertices of BT-tree
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What about RM/quadratic orbits?

- Suppose (for simplicity) that
  - $\Gamma^{\mathfrak{p}}$ is torsion-free and
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- Bieri-Eckmann duality:
$\rightsquigarrow \mathrm{H}^i(\Gamma_v, \mathrm{Div}\,\Gamma_v x) = \mathrm{H}_{d-i}(\Gamma_v, \mathrm{Div}\,\Gamma_v x)$

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Thank you

Merci