On quaternionic rigid meromorphic cocycles

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Motivation

Aim: generalize rigid meromorphic cocycles à la Darmon-Vonk

Q: What is a rigid meromorphic cocycle?

$$\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p) \quad p\text{-adic upper half plane}$$
$$\mathcal{M} = \text{ rigid meromorphic functions on } \mathcal{H}_p$$
$$\rightsquigarrow \mathrm{SL}_2(\mathbb{Z}[1/p]) \curvearrowright \mathcal{H}_p \quad \text{Möbius transformations}$$
$$\rightsquigarrow \mathrm{SL}_2(\mathbb{Z}[1/p]) \curvearrowright \mathcal{M}^\times$$

A: $J \in H^1_{par}(\mathrm{SL}_2(\mathbb{Z}[1/p]), \mathcal{M}^\times)$ rigid meromorphic cocycle

Theorem (Darmon-Vonk):

$$\operatorname{supp} \mathrm{Div}(J) \subseteq \text{finite union of RM-orbits}$$

More precise aim: generalize result to $\mathfrak{p}$-arithmetic subgroups of inner forms of SL_2 over arbitrary number fields

Basic idea I - a new approach

Study cohomology of $\mathrm{Div}^{\dagger}\,\mathcal{H}_p$ instead of $\mathcal{M}^{\times}$

$$\mathrm{Div}^{\dagger}\,\mathcal{H}_p = \text{ locally finite divisors}$$

Two ingredients:

1. description of $\mathrm{Div}^{\dagger}\,\mathcal{H}_p$ in terms of coinduction **and** induction
   - uses reduction map to Bruhat-Tits tree
2. Cohomological properties of arithmetic groups , i.e.:
   - Bieri-Eckmann duality
   - cohomology commutes with direct limits

Basic idea II - coordinate-free formulation

- F number field , B/F quaternion algebra that is split at $\mathfrak{p}$
- G = group of norm one elements of B

$$\rightsquigarrow G(F_{\mathfrak{p}}) \cong \mathrm{SL}_2(F_{\mathfrak{p}}) \quad \text{non-canonically!}$$
$$\rightsquigarrow G(F_{\mathfrak{p}}) \curvearrowright \mathcal{H}_{\mathfrak{p}} \quad \text{non-canonically!}$$

Solution: replace $\mathbb{P}^1$ by Brauer-Severi variety of B

$$\mathbb{P}_B(E) = \{I \lhd B \otimes_F E \text{ left ideal} \mid \dim_E I = 2\}$$

Brauer-Severi variety

$$\mathbb{P}_B(E) = \{I \triangleleft B \otimes_F E \text{ left ideal} \mid \dim_E I = 2\}$$

$$\mathbb{P}_B(E) \neq 0 \iff E \text{ splits } B$$

in that case: $\mathbb{P}_B(E) \cong \mathbb{P}^1(E)$ non-canonically!

put $\mathcal{H}_{B_\mathfrak{p}} = \mathbb{P}_B(\mathbb{C}_p) \setminus \mathbb{P}_B(F_\mathfrak{p})$

$\rightsquigarrow \mathcal{H}_{B_\mathfrak{p}} \cong \mathcal{H}_\mathfrak{p}$ non-canonically!

but $G(F_\mathfrak{p}) \curvearrowright \mathcal{H}_{B_\mathfrak{p}}$ **canonically!**

Finiteness of support

- $\Gamma^{\mathfrak{p}} \subseteq G(F)$ $\mathfrak{p}$-arithmetic subgroup
- $X \subseteq \mathcal{H}_{B_{\mathfrak{p}}}$ a $\Gamma^{\mathfrak{p}}$-stable subset

Lemma

The canonical map

$$\bigoplus_{\Gamma^{\mathfrak{p}} x \in \Gamma^{\mathfrak{p}} \backslash X} \mathrm{H}^i(\Gamma^{\mathfrak{p}}, \mathrm{Div}^{\dagger}\, \Gamma^{\mathfrak{p}} x) \longrightarrow \mathrm{H}^i(\Gamma^{\mathfrak{p}}, \mathrm{Div}^{\dagger}\, X)$$

is an isomorphism.

A bunch of reduction steps

- $o_1, \ldots, o_h$ the $\Gamma^{\mathfrak{p}}$-orbits of edges/vertices of BT-tree
$\rightsquigarrow X = \cup_{i=1}^{h} X_{o_h}$ with $X_{o_i} = X \cap \mathrm{red}^{-1}(o_i)$
$\rightsquigarrow \mathrm{Div}^\dagger X = \bigoplus_{i=1}^{h} \mathrm{Div}^\dagger X_{o_h}$ $\rightsquigarrow$ wlog $X = X_o$ for some orbit o

- pick $v \in o$, put $\Gamma_v = \mathrm{Stab}_{\Gamma^{\mathfrak{p}}} v$
$\rightsquigarrow \mathrm{Div}^\dagger X \cong \mathrm{Coind}_{\Gamma_v}^{\Gamma^{\mathfrak{p}}} \mathrm{Div}\, X_v$ with $X_v = X \cap \mathrm{red}^{-1}(v)$
$\rightsquigarrow \mathrm{H}^i(\Gamma^{\mathfrak{p}}, \mathrm{Div}^\dagger X) \cong \mathrm{H}^i(\Gamma_v, \mathrm{Div}\, X_v)$ by cohomological Shapiro

- $\mathrm{Div}\, X_v = \bigoplus_{\Gamma_v x \in \Gamma_v \backslash X_v} \mathrm{Div}\, \Gamma_v x$
$\rightsquigarrow \mathrm{H}^i(\Gamma_v, \mathrm{Div}\, X_v) \cong \bigoplus_{\Gamma_v x \in \Gamma_v \backslash X_v} \mathrm{H}^i(\Gamma_v, \mathrm{Div}\, \Gamma_v x)$
since arithmetic groups are of type (VFL)

- coinduce back to $\Gamma^{\mathfrak{p}}$

What about RM/quadratic orbits?

- Suppose (for simplicity) that
  - $\Gamma^{\mathfrak{p}}$ is torsion-free and
  - B is non-split

- Bieri-Eckmann duality:
$\rightsquigarrow \mathrm{H}^i(\Gamma_v, \mathrm{Div}\,\Gamma_v x) = \mathrm{H}_{d-i}(\Gamma_v, \mathrm{Div}\,\Gamma_v x)$

- homological Shapiro:
$\rightsquigarrow \mathrm{H}_{d-i}(\Gamma_v, \mathrm{Div}\,\Gamma_v x) = \mathrm{H}_{d-i}(\Gamma_x, \mathbb{Z})$ with $\Gamma_x = \mathrm{Stab}_{\Gamma_v} x$

- $\mathrm{H}_i(\Gamma_x, \mathbb{Z}) \neq 0$ for some $i > 0$
iff Γ_x is non-trivial
iff x is defined over quadratic non-CM extension of F

Thank you
Merci