L-INVARIANTS FOR COHOMOLOGICAL REPRESENTATIONS OF PGL(2) OVER ARBITRARY NUMBER FIELDS

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Abstract. Let π be a cuspidal, cohomological automorphic representation of an inner form G of PGL_2 over a number field F of arbitrary signature. Further, let $\mathfrak{p}$ be a prime of F such that G is split at $\mathfrak{p}$ and the local component $\pi_{\mathfrak{p}}$ of π at $\mathfrak{p}$ is the Steinberg representation. Assuming that the representation is non-critical at $\mathfrak{p}$ we construct automorphic $\mathcal{L}$-invariants for the representation π. If the number field F is totally real, we show that these automorphic $\mathcal{L}$-invariants agree with the Fontaine–Mazur $\mathcal{L}$-invariant of the associated p-adic Galois representation. This generalizes a recent result of Spieß respectively Rosso and the first named author from the case of parallel weight 2 to arbitrary cohomological weights.

Contents

Introduction

The purpose of this article is threefold: first, let π be a cuspidal, cohomological automorphic representation of an inner form G of PGL_2 over a number field F of arbitrary signature and $\mathfrak{p}$ be a prime of F such that G is split at $\mathfrak{p}$ and the local component $\pi_{\mathfrak{p}}$ of π at $\mathfrak{p}$ is the Steinberg representation. We want to give a general construction of automorphic $\mathcal{L}$-invariants (also known as Teitelbaum, Darmon or Orton $\mathcal{L}$-invariants) for π. For representations which are cohomological with respect to the trivial coefficient system, or in other words for forms of parallel weight 2, these $\mathcal{L}$-invariants have been defined in general (see for example [21]) but for higher weights they have been defined only in certain situations:

1. In case $F=\mathbb{Q}$ and G is compact at infinity by Teitelbaum in [36],
2. in case $F=\mathbb{Q}$ and G is split by Orton in [29],
3. in case $F=\mathbb{Q}$ and G is split at infinity by Rotger and Seveso in [30],
4. in case F is imaginary quadratic and G is split by Barrera-Salazar and Williams in [2] and
5. in case F is totally real and G is compact at infinity by Chida, Mok and Park in [12].

An obstacle that might have prevented the construction in general is the following: whereas in case of the trivial coefficient system the representation is always ordinary and therefore non-critical at $\mathfrak{p}$, this is no longer true for higher weights. But note that for the cases 1.-4. above the representation has still non-critical slope at $\mathfrak{p}$

and, thus, is non-critical at $\mathfrak{p}$ (see the end of Section 2.4 for a detailed discussion on non-critical slopes). It seems to the authors of this paper that in [12] it is implicitly assumed that the representation is non-critical at $\mathfrak{p}$ (see Remark 2.6 for more details). Our first main result (see Definition 3.2) is the construction of automorphic $\mathcal{L}$-invariants under the assumption that the representation π is non-critical at $\mathfrak{p}$. We point out that our construction of $\mathcal{L}$-invariants is novel as it does not involve the Bruhat-Tits tree at any stage.

Our second goal is to bridge the gap between works using overconvergent cohomology à la Ash-Stevens, for example [2] and [3], and Spieß' more representation-theoretic approach (cf. [34]). In particular, we show that the non-criticality condition for classes in overconvergent cohomology that is discussed in [3] respectively [5] is equivalent to a more representation-theoretic one: assume for the moment that $\pi_{\mathfrak{p}}$ is not necessarily Steinberg but merely has an Iwahori-fixed vector. we explain that choosing a $\mathfrak{p}$-stabilization of π, i.e., an Iwahori-fixed vector of $\pi_{\mathfrak{p}}$, yields a cohomology class of a $\mathfrak{p}$-arithmetic subgroup of $G(F)$ with values in the dual of a locally algebraic principal series representation of $G(F_{\mathfrak{p}})$. Non-criticality is then equivalent to the fact that this class lifts uniquely to a class in the cohomology with values in the continuous dual of the corresponding locally analytic principal series representation (see Proposition 2.13). The main tool to prove this equivalence is the resolution of locally analytic principal series representations by Kohlhaase and Schraen (see [27]).

Finally, we show that, if the number field F is totally real the automorphic $\mathcal{L}$-invariants attached to π agree with the derivatives of the $U_{\mathfrak{p}}$-eigenvalue of a p-adic family passing through π (cf. Theorem 4.3). This equality is known in case $F = \mathbb{Q}$ by the work of Bertolini-Darmon-Iovita (see [6]) and Seveso (see [33]). For Hilbert modular forms of parallel weight 2 the equality was recently proven independently by Rosso and the first named author (see [23]). As we do not work with general reductive groups as in *loc.cit.* the arguments simplify substantially making them more accessible to people who are only interested in Hilbert modular forms. Furthermore, it is known that the derivatives of the $U_{\mathfrak{p}}$-eigenvalue agree with the Fontaine–Mazur $\mathcal{L}$-invariant of the associated Galois representation, if that Galois representation is non-critical. Thus, we deduce the equality of automorphic and Fontaine–Mazur $\mathcal{L}$-invariants (see [35] for an independent proof of this equality in case of parallel weight 2). The equality of $\mathcal{L}$-invariants in parallel weight 2 is necessary for the construction of plectic Stark-Heegner points in recent work of Fornea and the first named author (cf. [18]). This article should be seen as a precursor for defining plectic Stark-Heegner cycles for arbitrary cohomological weights.

Notations. All rings will be commutative and unital. The units of a ring R will be denoted by $R^\times$. If R is a ring and G is a group, we denote the group algebra of G over R by $R[G]$. The trivial character of any group will be denoted by $\mathbb{1}$. Given two sets X and Y we will write $\mathcal{F}(X, Y)$ for the set of all maps from X to Y. If X and Y are topological spaces, we write $C(X, Y) \subseteq \mathcal{F}(X, Y)$ for the set of all continuous maps. If Y is a topological group, we denote by $C_c(X, Y) \subseteq C(X, Y)$ the subset of functions with compact support.

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Setup. *A number field.* We fix an algebraic number field $F \subseteq \mathbb{C}$ with ring of integers $\mathcal{O}_F$. We write S_∞ for the set of infinite places of F and Σ for the set of all embeddings from F into the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. The action of complex conjugation on Σ will be denoted by c. We write δ for the number of complex places of F.

For any place $\mathfrak{q}$ of F we will denote by $F_\mathfrak{q}$ the completion of F at $\mathfrak{q}$. If $\mathfrak{q}$ is a finite place, we let $\mathcal{O}_\mathfrak{q}$ denote the valuation ring of $F_\mathfrak{q}$ and $\mathrm{ord}_\mathfrak{q}$ the additive valuation such that $\mathrm{ord}_\mathfrak{q}(\varpi) = 1$ for any local uniformizer $\varpi \in \mathcal{O}_\mathfrak{q}$. We write $\mathcal{N}(\mathfrak{q})$ for the cardinality of the residue field $\mathcal{O}/\mathfrak{q}$. We normalize the $\mathfrak{q}$-adic absolute valuation $|\cdot|_\mathfrak{q}$ by $|\varpi|_\mathfrak{p} = \mathcal{N}(\mathfrak{q})^{-1}$.

For a finite set S of places of F we define the "S-truncated adeles" $\mathbb{A}^S$ as the restricted product of all completions F_v with $v \notin S$. In case S is the empty set we drop the superscript S from the notation. We will often write $\mathbb{A}^{S,\infty}$ instead of $\mathbb{A}^{S\cup S_\infty}$.

If H is an algebraic group over F, we will put $H_\mathfrak{q} = H(F_\mathfrak{q})$ for any place $\mathfrak{q}$ of F. If S is a finite set of places of F, we will write $H_S = \prod_{\mathfrak{q}\in S} H_\mathfrak{q}$. Further, we abbreviate $H_\infty = H_{S_\infty}$.

A quaternion algebra. We fix a quaternion algebra D over F. We denote by $\mathrm{ram}(D)$ the set of places of F at which D is ramified and put

$$\mathrm{disc}(D) = \prod_{\mathfrak{q}\in\mathrm{ram}(D),\mathfrak{q}\nmid\infty} \mathfrak{q}.$$

Let $D^\times$ be the group of units of D considered as an algebraic group over F. The centre $Z \subseteq D^\times$ is naturally isomorphic to the multiplicative group $\mathbb{G}_m$. We put $G = D^\times/Z$. For any place $\mathfrak{q} \notin \mathrm{ram}(D)$ we fix an isomorphism $D_\mathfrak{q} \cong M_2(F_\mathfrak{q})$ that in turn induces an isomorphism $G_\mathfrak{q} \cong \mathrm{PGL}_2(F_\mathfrak{q})$. For any Archimedean place $\mathfrak{q} \in \mathrm{ram}(D)$ we fix an isomorphism of $D_\mathfrak{q}$ with the Hamilton quaternions, which yields an embedding $G_\mathfrak{q} \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$. In particular, we get an injection $j_\sigma \colon G(F) \subseteq G(F_\mathfrak{q}) \xhookrightarrow{\sigma_*} \mathrm{PGL}_2(\mathbb{C})$ for every embedding $\sigma \in \Sigma$ with underlying place $\mathfrak{q}$. We write

$$j\colon G(F) \longhookrightarrow \prod_{\sigma\in\Sigma} \mathrm{PGL}_2(\mathbb{C})$$

for the diagonal embedding.

Let $S_\infty(D)$ be the set of all Archimedean places of F at which D is split. We put

$$q = \#S_\infty(D).$$

Let $S_\mathbb{R}(D) \subseteq S_\infty(D)$ be the subset of real places. We denote by G_∞^+ the connected component of the identity of G_∞ . The group $\mathrm{PGL}_2(\mathbb{C})$ and the units of the Hamilton quaternions are connected, whereas $\mathrm{PGL}_2(\mathbb{R})$ has two connected components. Therefore, we can identify

$$\pi_0(G_\infty) = G_\infty/G_\infty^+ \cong \{\pm 1\}^{S_\mathbb{R}(D)}$$

If $A \subseteq G_\infty$ is a subgroup, we put $A^+ = A \cap G_\infty^+$.

An automorphic representation. Let $\pi' = \otimes_\mathfrak{q} \pi'_\mathfrak{q}$ be a cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$ that is cohomological (see Section 1.1.2). If F is totally real, then such automorphic representations (up to twists by the norm character) are in one-to-one correspondence with cuspidal Hilbert modular newforms with even weights and trivial Nebentypus.

We assume that the local component $\pi'_\mathfrak{q}$ is either a twist of the Steinberg representation or supercuspidal for all $\mathfrak{q}$ dividing $\mathrm{disc}(D)$. Thus, there exists a Jacquet–Langlands transfer π of π' to $G(\mathbb{A})$, i.e. an automorphic representation of $G(\mathbb{A})$ such that $\pi'_\mathfrak{q} \cong \pi_\mathfrak{q}$ for all places $\mathfrak{q} \notin \mathrm{ram}(D)$. Moreover, we have that $\pi_\mathfrak{q}$ is one-dimensional for all $\mathfrak{q} \mid \mathrm{disc}(D)$ such that $\pi'_\mathfrak{q}$ is a twist of the Steinberg representation.

1. Cohomology of p-arithmetic groups

We recollect some basic facts about the cohomology of $\mathfrak{p}$-arithmetic groups with values in duals of smooth representations.

1.1. **The Eichler–Shimura isomorphism.**

In this section we recall how the representation π contributes to the cohomology of the locally symmetric space attached to G.

1.1.1. *Weights and coefficient modules.* Let $k \geq 0$ be an even integer. For any ring R we let $V_k(R) \subseteq R[X,Y]$ be the space of homogeneous polynomials of degree k with $\mathrm{PGL}_2(R)$-action given by

$$(g.f)(X,Y) = \det(g)^{-k/2} f(bY + dX, aY + cX) \text{ for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(R).$$

We may attach to $f \in V_k(R)$ an R-valued function on G via $\psi_f(g) = (g.f)(1,0)$. In case $b = \begin{pmatrix} b_1 & u \\ 0 & b_2 \end{pmatrix}$ is an upper triangular matrix, we have

$$\begin{aligned} \psi_f(bg) = (bg.f)(1,0) &= (b.(g.f))(1,0) \\ &= b_1^{-k/2} b_2^{k/2} \cdot (g.f)(1,0) \\ &= b_1^{-k/2} b_2^{k/2} \cdot \psi_f(g) \end{aligned}$$

for all $g \in G$.

For a *weight* $\boldsymbol{k} = (k_\sigma)_{\sigma\in\Sigma} \in 2\mathbb{Z}_{\geq 0}[\Sigma]$ we define the $\mathrm{PGL}_2(R)^\Sigma$-representation

$$V_{\boldsymbol{k}}(R) = \otimes_{\sigma\in\Sigma} V_{k_\sigma}(R).$$

and $V_{\boldsymbol{k}}(R)^\vee$ as the R-linear dual of $V_{\boldsymbol{k}}(R)$.

We may view $V_{\boldsymbol{k}}(\mathbb{C})$ as a representation of $G(F)$ via the embedding j. In fact, there exists a number field $E \subseteq \mathbb{C}$ such that every embedding $\sigma\colon F \hookrightarrow \mathbb{C}$ factors over E and such that $G(F)$ acts on $V_{\boldsymbol{k}}(E) \subseteq V_{\boldsymbol{k}}(\mathbb{C})$.

1.1.2. $(\mathfrak{g}, K_\infty)$-*cohomology.* Let $\mathfrak{g}$ denote the complexification of the Lie-Algebra of G_∞. We fix a maximal compact subgroup K_∞ of G_∞ with connected component

We recall that π is cohomological if and only if there exists a weight $\boldsymbol{k} = (k_\sigma)_{\sigma\in\Sigma}$ such that

$$\mathrm{H}^\bullet(\mathfrak{g}, K_\infty^+, \pi_\infty \otimes V_{\boldsymbol{k}}(\mathbb{C})^\vee) \neq 0.$$

See [7] for the notion of $(\mathfrak{g}, K_\infty)$-cohomology.

The weight $\boldsymbol{k}$ is unique and we fix it from here on. By [25] (3.6.1) we know that

$$k_\sigma = k_{c\sigma} \tag{1.1}$$

holds for all $\sigma \in \Sigma$. The group $\pi_0(G_\infty) \cong \pi_0(K_\infty)$ acts on $(\mathfrak{g}, K_\infty)$-cohomology. For every character $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ we have the following dimension formulas for the ϵ-isotypic component:

$$\dim_{\mathbb{C}} \mathrm{H}^i(\mathfrak{g}, K_\infty^+, \pi_\infty \otimes V_{\boldsymbol{k}}^\vee)^\epsilon = \binom{\delta}{i-q}. \tag{1.2}$$

Via the Künneth theorem one may reduce the computation to that of the cohomology for each place $\mathfrak{q} \mid \infty$ separately. In case $G_\mathfrak{q}$ is split, the computation is spelled out in Section (3.6.2) of [25]. The non-split case is trivial.

1.1.3. *Local systems.* For an open compact subgroup $K \subseteq G(\mathbb{A}^\infty)$ we define the locally symmetric space of level K as

$$\mathcal{X}_K = G(F)\backslash G(\mathbb{A})/KK_\infty^+.$$

If K is small enough, the topological space $\mathcal{X}_K$ carries the structure of a smooth manifold.

We fix a field Ω of characteristic zero. Let N be an $\Omega[G(F)]$-module and $\underline{N}$ the locally constant sheaf on $\mathcal{X}_K$ given by the fibres of the projection

$$G(F)\backslash G(\mathbb{A}) \times N/KK_\infty^+ \twoheadrightarrow \mathcal{X}_K,$$

where we let $G(F)$ act on $G(\mathbb{A}) \times N$ diagonally.

Right translation defines commuting actions of the component group $\pi_0(G_\infty)$ and the Hecke algebra $\mathbb{T}_K(\Omega) = \Omega[K\backslash G(\mathbb{A}^\infty)/K]$ of level K on N-valued cohomology $\mathrm{H}^\bullet(\mathcal{X}_K, \underline{N})$.

We assume in the following that K is of the form $K = \prod_{\mathfrak{q}} K_{\mathfrak{q}}$ and that

$$(\pi^\infty)^K \neq 0.$$

Let S be a finite set of primes of F such that S contains every $\mathfrak{q}$ such that $K_{\mathfrak{q}}$ is not maximal and put $K^S = \prod_{\mathfrak{q}\notin S} K_{\mathfrak{q}}$. The Hecke algebra

$$\mathbb{T}^S_{K^S}(\Omega) = \Omega[K^S\backslash G(\mathbb{A}^{S,\infty})/K^S]$$

away from S is central in $\mathbb{T}_K(\Omega)$.

We will assume for the rest of this article that π^∞ has a model over E, which we denote π^∞_E. We may always assume this by enlarging E slightly (see [26], Theorem C). If Ω is an extension of E, we put $\pi^\infty_\Omega = \pi^\infty_E \otimes_E \Omega$. Let $\mathfrak{m}^S_\pi \subseteq \mathbb{T}^S_{K^S}(\Omega)$ be the maximal ideal that is given by the kernel of the map $\mathbb{T}^S_{K^S}(\Omega) \to \mathrm{End}_\Omega((\pi^\infty_\Omega)^K)$.

Theorem 1.1. *Let Ω be any extension of E. For every character $\epsilon\colon \pi_0(G_\infty) \to \{\pm 1\}$ we have*

$$\dim_\Omega \mathrm{Hom}_{\mathbb{T}_K(\Omega)}((\pi^\infty_\Omega)^K, \mathrm{H}^i(\mathcal{X}_K, \underline{V_{\boldsymbol{k}}(\Omega)^\vee})^\epsilon) = \binom{\delta}{i-q}.$$

Moreover, the localization $\mathrm{H}^i(\mathcal{X}_K, \underline{V_{\boldsymbol{k}}(\Omega)^\vee})_{\mathfrak{m}^S_\pi}$ is equal to the sum of the images of all homomorphisms from $(\pi^\infty_\Omega)^K$ to $\mathrm{H}^i(\mathcal{X}_K, \underline{V_{\boldsymbol{k}}(\Omega)^\vee})$.

Proof. One may deduce from the Borel-Serre compactification that the canonical map

$$\mathrm{H}^i(\mathcal{X}_K, \underline{V_{\boldsymbol{k}}(E)^\vee})^\epsilon \otimes_E \Omega \longrightarrow \mathrm{H}^i(\mathcal{X}_K, \underline{V_{\boldsymbol{k}}(\Omega)^\vee})^\epsilon$$

is an isomorphism for every extensions Ω of E. Thus, we may reduce to the case $\Omega = \mathbb{C}$. In that case, the first claim follows from standard arguments about cohomological representations and equation (1.2) (see for example Section III of [25] for details in the case G is split).

The second claim follows from strong multiplicity one. □

1.1.4. *Cohomology of arithmetic groups.* We are going to recast the above cohomology groups in terms of group cohomology. Let $K \subseteq G(\mathbb{A}^\infty)$ be an open compact subgroup and N an $\Omega[G(F)]$-module. The group $G(F)$ acts on the space $\mathcal{F}(G(\mathbb{A}^\infty)/K, N)$ via $(\gamma.f)(g) = \gamma.f(\gamma^{-1}g)$ and the Hecke algebra $\mathbb{T}_K(\Omega)$ via right translation. Thus we have commuting actions of the component group $\pi_0(G_\infty)$ and the Hecke algebra $\mathbb{T}_K(\Omega) = \Omega[K\backslash G(\mathbb{A}^\infty)/K]$ on the spaces

$$\mathrm{H}^i(X_K, N) := \mathrm{H}^i(G(F)^+, \mathcal{F}(G(\mathbb{A}^\infty)/K, N))$$

A straightforward calculation shows the following:

Lemma 1.2. *Suppose that N is a $\mathbb{Q}$-vector space. There are canonical isomorphisms*

$$\mathrm{H}^\bullet(X_K, N) \xrightarrow{\cong} \mathrm{H}^\bullet(\mathcal{X}_K, \underline{N})$$

that are equivariant with respect to the actions of the component group and the Hecke algebra.

1.2. **Cohomology of p-arithmetic groups.** Let $\mathfrak{p}$ be a prime of F and Ω a field of characteristic zero. Given an open compact subgroup $K^{\mathfrak{p}} \subseteq G(\mathbb{A}^{\mathfrak{p},\infty})$, an $\Omega[G_{\mathfrak{p}}]$-module M and an $\Omega[G(F)]$-module N we let $G(F)$ act on the Ω-vector space $\mathcal{F}(G(\mathbb{A}^{\mathfrak{p},\infty})/K^{\mathfrak{p}}, \mathrm{Hom}_\Omega(M,N))$ via $(\gamma.f)(g)(m) = \gamma.f(\gamma^{-1}g)(\gamma^{-1}(m))$ and put

$$\mathrm{H}^i_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, M, N) := \mathrm{H}^i(G(F)^+, \mathcal{F}(G(\mathbb{A}^{\mathfrak{p},\infty})/K^{\mathfrak{p}}, \mathrm{Hom}_\Omega(M,N))).$$

Again one can define commuting actions of the component group $\pi_0(G_\infty)$ and the Hecke algebra

$$\mathbb{T}^{\mathfrak{p}}_{K^{\mathfrak{p}}}(\Omega) = \Omega[K^{\mathfrak{p}} \backslash G(\mathbb{A}^{\mathfrak{p},\infty})/K^{\mathfrak{p}}].$$

For later purposes we also define the following continuous variant: let A be an affinoid $\mathbb{Q}_p$-algebra. Given a continuous A-module M with a continuous $G_{\mathfrak{p}}$-action and an $A[G(F)]$-module N that is finitely generated and free over A we put

$$\mathrm{H}^i_{A,\mathrm{cont}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, M, N) := \mathrm{H}^i(G(F)^+, \mathcal{F}(G(\mathbb{A}^{\mathfrak{p},\infty})/K^{\mathfrak{p}}, \mathrm{Hom}_{A,\mathrm{cont}}(M,N))).$$

Suppose that $M = M_1 \otimes_\Omega M_2$ with both M_1 and M_2 being $\Omega[G_{\mathfrak{p}}]$-modules. Then by definition we have

$$\mathrm{H}^i_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, M_1 \otimes M_2, N) = \mathrm{H}^i_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, M_1, \mathrm{Hom}_\Omega(M_2, N)), \tag{1.3}$$

where $G(F)$ acts on M_2 via the embedding $G(F) \hookrightarrow G_{\mathfrak{p}}$.

Let us discuss an example of the above construction. First we are going to recall the notion of compact induction: let $K_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ be an open compact subgroup and L a $\Omega[K_{\mathfrak{p}}]$-module. The *compact induction* $\text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} L$ of L to $G_{\mathfrak{p}}$ is given by the set of functions $f\colon G_{\mathfrak{p}} \to L$ that satisfy $f(gk) = k^{-1}.f(g)$ for all $g \in G_{\mathfrak{p}}$, $k \in K_{\mathfrak{p}}$ and have finite support modulo $K_{\mathfrak{p}}$. The group $G_{\mathfrak{p}}$ acts on $\text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} L$ via left translation.

If L is a $\Omega[G_{\mathfrak{p}}]$-module, the map

$$L \otimes_\Omega \text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega \longrightarrow \text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} L,\ l \otimes f \longmapsto [g \mapsto f(g) \cdot g^{-1}.l] \tag{1.4}$$

is a $G_{\mathfrak{p}}$-equivariant isomorphism. Thus in this case we get a canonical $\mathbb{T}^{\mathfrak{p}}_{K^{\mathfrak{p}}}(\Omega)$-equivariant isomorphisms

$$\mathrm{H}^i_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} L, N) \cong \mathrm{H}^i(X_{K^{\mathfrak{p}} \times K_{\mathfrak{p}}}, \mathrm{Hom}_\Omega(L, N)), \tag{1.5}$$

by (1.3).

We define

$$\tilde{\mathrm{H}}^i(X_{K^{\mathfrak{p}}}, N) = \varinjlim_{K_{\mathfrak{p}}} \mathrm{H}^i(X_{K^{\mathfrak{p}} \times K_{\mathfrak{p}}}, N),$$

where the limit is taken over all open compact subgroups $K_{\mathfrak{p}} \subseteq G_{\mathfrak{p}}$. This space carries commuting actions of $\pi_0(G_\infty)$, $G_{\mathfrak{p}}$ and $\mathbb{T}^{\mathfrak{p}}_{K^{\mathfrak{p}}}(\Omega)$. The goal of this section is to compare $\mathrm{Hom}_{\Omega[G_{\mathfrak{p}}]}(M, \tilde{\mathrm{H}}^i(X_{K^{\mathfrak{p}}}, N))$ and $\mathrm{H}^i_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, M, N)$ in certain situations.

Lemma 1.3. *Let M be a smooth $\Omega[G_{\mathfrak{p}}]$-representation of finite length. Then, there exists a resolution*

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_i, $i = 0, 1$, are finitely generated projective smooth representations.

Proof. In case $\mathfrak{p} \in \mathrm{ram}(D)$ the group $G_{\mathfrak{p}}$ is compact. Thus, the category of smooth representations of $G_{\mathfrak{p}}$ with Ω-coefficients is semi-simple and M is projective itself.

In case D is split at $\mathfrak{p}$, and therefore $G_{\mathfrak{p}} \cong \mathrm{PGL}_2(F_{\mathfrak{p}})$, this is a consequence of the main theorem of [32]. □

Let $\pi^{\mathfrak{p},\infty}$ be the restricted tensor product of all local components of π away from $\mathfrak{p}$ and ∞. This is a smooth irreducible representation of $G(\mathbb{A}^{\mathfrak{p},\infty})$. We fix models $\pi_E^{\mathfrak{p},\infty}$ respectively $\pi_{\mathfrak{p},E}$ of $\pi^{\mathfrak{p},\infty}$ respectively $\pi_{\mathfrak{p}}$ over E and a $G(\mathbb{A}^\infty)$-equivariant isomorphism

$$\pi_E^{\mathfrak{p},\infty} \otimes_E \pi_{\mathfrak{p},E} \cong \pi_E^\infty.$$

From now on Ω will always be an extension of E and we put $\pi_\Omega^{\mathfrak{p},\infty} = \pi_E^{\mathfrak{p},\infty} \otimes_E \Omega$ as well as $\pi_{\mathfrak{p},\Omega} = \pi_{\mathfrak{p},E} \otimes_E \Omega$. Further, we assume that $K^{\mathfrak{p}}$ is of the form $K^{\mathfrak{p}} = \prod_{\mathfrak{q}\neq\mathfrak{p}} K_{\mathfrak{q}}$ and that $(\pi_\Omega^{\mathfrak{p},\infty})^{K^{\mathfrak{p}}} \neq 0$. We fix a finite set S of primes of F as before such that $\mathfrak{p} \in S$.

Proposition 1.4. *Let M be a smooth $\Omega[G_{\mathfrak{p}}]$-representation of finite length. There are isomorphisms*

$$\mathrm{H}^q_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, M, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \xrightarrow{\cong} \mathrm{Hom}_{\Omega[G_{\mathfrak{p}}]}(M, \tilde{\mathrm{H}}^q(X_{K^{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi})$$

that are functorial in M and equivariant under the actions of $\mathbb{T}^{\mathfrak{p}}_{K^{\mathfrak{p}}}(\Omega)$ and $\pi_0(G_\infty)$. Furthermore, we have

$$\dim_\Omega \mathrm{H}^d_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, M, V_{\boldsymbol{k}}(\Omega)^\vee) < \infty$$

for all $d \geq 0$.

Proof. If $M = P$ is projective, there are functorial isomorphisms

$$\mathrm{H}^d_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, P, V_{\boldsymbol{k}}(\Omega)^\vee) \xrightarrow{\cong} \mathrm{Hom}_{\Omega[G_{\mathfrak{p}}]}(P, \tilde{\mathrm{H}}^d(X_{K^{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^\vee)) \tag{1.6}$$

for all $d \geq 0$ by [22], Lemma 3.5 (b). In particular, we have

$$\mathrm{H}^d_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, P, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} = 0$$

for all $d < q$ by Theorem 1.1.

Thus the short exact sequence

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

of Lemma 1.3 induces the exact sequence

$$0 \to \mathrm{H}^q_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, M, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \to \mathrm{H}^q_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, P_0, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \to \mathrm{H}^q_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, P_1, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi}$$

and the first claim follows from the isomorphism (1.6) for $P = P_0, P_1$.

The second claim follows similarly. ☐

Remark 1.5. A typical projective smooth representation is the compact induction $\text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega$ of the trivial representation from an open compact subgroup $K_{\mathfrak{p}} \subseteq G_{\mathfrak{p}}$. In that case the result of [22] used above simply follows from the composition of isomorphisms

$$\begin{aligned}\mathrm{H}^d_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega, N) &\cong \mathrm{H}^d_\Omega(X_{K^{\mathfrak{p}}\times K_{\mathfrak{p}}}, N)\\ &\cong \tilde{\mathrm{H}}^d(X_{K^{\mathfrak{p}}}, N)^{K_{\mathfrak{p}}}\\ &\cong \mathrm{Hom}_{\Omega[G_{\mathfrak{p}}]}(\text{c-ind}_{K_{\mathfrak{p}}}^{G_{\mathfrak{p}}} \Omega, \tilde{\mathrm{H}}^d(X_{K^{\mathfrak{p}}}, N)^{K_{\mathfrak{p}}}),\end{aligned}$$

where the first isomorphism is a special case of (1.5), the second follows from the Hochschild-Serre spectral sequence and the third from Frobenius reciprocity.

Proposition 1.4 together with Theorem 1.1 implies the following:

Corollary 1.6. *Let M be an irreducible smooth $\Omega[G_{\mathfrak{p}}]$-representation. Then*

$$\mathrm{H}^q_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, M, V_{\boldsymbol{k}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi} = 0$$

unless $M \cong \pi_{\mathfrak{p},\Omega}$. Furthermore, there is an isomorphism

$$\mathrm{H}^q_\Omega(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \pi_{\mathfrak{p},\Omega}, V_{\boldsymbol{k}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi} \cong (\pi_\Omega^{\mathfrak{p},\infty})^{K_{\mathfrak{p}}}$$

of $\mathbb{T}^{\mathfrak{p}}_{K^{\mathfrak{p}}}(\Omega)$*-modules. In particular, it is an absolutely irreducible* $\mathbb{T}^{\mathfrak{p}}_{K^{\mathfrak{p}}}(\Omega)$*-module.*

Remark 1.7. The corollary above was implicitly proven in [34] under the assumption that the local representation $\pi_{\mathfrak{p}}$ has an Iwahori-fixed vector via computations with explicit resolution of $\pi_{\mathfrak{p}}$ coming from the Bruhat–Tits tree.

1.3. **The Steinberg case.** We now assume that D is split at $\mathfrak{p}$. We define the Ω*-valued smooth Steinberg representation* $\mathrm{St}^{\infty}_{\mathfrak{p}}(\Omega)$ of $G_{\mathfrak{p}}$ as the space of locally constant Ω-valued functions on $\mathbb{P}^1(F_{\mathfrak{p}})$ modulo constant functions. The group $G_{\mathfrak{p}} \cong \mathrm{PGL}_2(F_{\mathfrak{p}})$ naturally acts on $\mathbb{P}^1(F_{\mathfrak{p}})$ and thus also on $\mathrm{St}^{\infty}_{\mathfrak{p}}(\Omega)$.

We assume for the moment that $\pi_{\mathfrak{p}} = \mathrm{St}^{\infty}_{\mathfrak{p}}(\mathbb{C})$. Then by Corollary 1.6 we know that $\mathrm{H}^q_{\Omega}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathrm{St}^{\infty}_{\mathfrak{p}}(\Omega), V_{\boldsymbol{k}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}^S_{\pi}}$ is an irreducible $\mathbb{T}^{\mathfrak{p}}_{K^{\mathfrak{p}}}(\Omega)$-module.

Given smooth Ω-representations V and W of $G_{\mathfrak{p}}$ we denote by $\mathrm{Ext}^i_{\infty}(V, W)$ the Ext-groups in the category of smooth representations. It is well known that

$$\dim_{\Omega} \mathrm{Ext}^i_{\infty}(\Omega, \mathrm{St}^{\infty}_{\mathfrak{p}}(\Omega)) = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.7}$$

and

$$\dim_{\Omega} \mathrm{Ext}^i_{\infty}(\mathrm{St}^{\infty}_{\mathfrak{p}}(\Omega), \mathrm{St}^{\infty}_{\mathfrak{p}}(\Omega)) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

The above calculations follow directly from the existence of the following two projective resolutions:

$$0 \longrightarrow \text{c-ind}^{G_{\mathfrak{p}}}_{J_{\mathfrak{p}}} \chi_- \longrightarrow \text{c-ind}^{G_{\mathfrak{p}}}_{K_{\mathfrak{p}}} \Omega \longrightarrow \Omega \longrightarrow 0$$

and

$$0 \longrightarrow \text{c-ind}^{G_{\mathfrak{p}}}_{K_{\mathfrak{p}}} \Omega \longrightarrow \text{c-ind}^{G_{\mathfrak{p}}}_{J_{\mathfrak{p}}} \chi_- \longrightarrow \mathrm{St}^{\infty}_{\mathfrak{p}}(\Omega) \longrightarrow 0.$$

Here $K_{\mathfrak{p}} = \mathrm{PGL}_2(\mathcal{O}_{\mathfrak{p}})$ is a maximal compact subgroup, $J_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ is an open compact subgroup that contains an Iwahori subgroup as a normal subgroup and $\chi_- \colon J_{\mathfrak{p}} \to \{\pm 1\}$ is a non-trivial character that is trivial on the Iwahori subgroup. The exactness of the first sequence is just a reformulation of the contractibility of the Bruhat–Tits tree. The second follows from the fact that the cohomology with compact support of the Bruhat–Tits tree is the Steinberg representation (see for example [34], equation (18)). To keep with our promise not to use the Bruhat-Tits tree let us mention that alternative calculations of these Ext-groups that do not invoke the Bruhat–Tits tree - and work for more general reductive groups - can be found in [28] and [13].

Let $\mathcal{E}^{\infty}$ be the unique (up to multiplication by a scalar) extension of Ω by $\mathrm{St}^{\infty}_{\mathfrak{p}}(\Omega)$, i.e., there exists a non-split exact sequence

$$0 \longrightarrow \mathrm{St}^{\infty}_{\mathfrak{p}}(\Omega) \longrightarrow \mathcal{E}^{\infty} \longrightarrow \Omega \longrightarrow 0. \tag{1.9}$$

of $G_{\mathfrak{p}}$-modules.

We denote by

$$c^{\epsilon}_{\infty} \colon \mathrm{H}^q_{\Omega}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathrm{St}^{\infty}_{\mathfrak{p}}(\Omega), V_{\boldsymbol{k}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}^S_{\pi}} \longrightarrow \mathrm{H}^{q+1}_{\Omega}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \Omega, V_{\boldsymbol{k}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}^S_{\pi}}$$

the (localization of the) boundary homomorphism of the long exact sequence induced by (1.9). It is a homomorphism of $\mathbb{T}^{\mathfrak{p}}_{K^{\mathfrak{p}}}(\Omega)$-modules.

The following generalization of [34], Lemma 6.2 (b) holds:

Lemma 1.8. *The map*

$$c^{\epsilon}_{\infty} \colon \mathrm{H}^d_{\Omega}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathrm{St}^{\infty}_{\mathfrak{p}}(\Omega), V_{\boldsymbol{k}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}^S_{\pi}} \longrightarrow \mathrm{H}^{d+1}_{\Omega}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \Omega, V_{\boldsymbol{k}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}^S_{\pi}}$$

is an isomorphism for every sign character ϵ *and every* $d \geq 0$.

Proof. We have to show that

$$\mathrm{H}^d_\Omega(X^{\mathfrak{p}}_{K^\mathfrak{p}}, \mathcal{E}^\infty, V_{\boldsymbol{k}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi} = 0$$

for all $d \geq 0$. Let

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathcal{E}^\infty \longrightarrow 0 \tag{1.10}$$

be a projective resolution of $\mathcal{E}^\infty$ as in Lemma 1.3.

Again by [22], Lemma 3.5 (b) we have

$$\mathrm{H}^d_\Omega(X^{\mathfrak{p}}_{K^\mathfrak{p}}, P_i, V_{\boldsymbol{k}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi} \xrightarrow{\cong} \mathrm{Hom}_{\Omega[G_\mathfrak{p}]}(P_i, \tilde{\mathrm{H}}^d(X_{K^\mathfrak{p}}, V_{\boldsymbol{k}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi})$$

for all $d \geq 0$ and $i = 0, 1$. As a $G_\mathfrak{p}$-module $\tilde{\mathrm{H}}^d(X_{K^\mathfrak{p}}, V_{\boldsymbol{k}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi}$ is isomorphic to some copies of $\mathrm{St}^\infty_\mathfrak{p}(\Omega)$ by our assumption and Theorem 1.1.

Thus, analyzing the long exact sequence induced from (1.10) it is enough to show that

$$\mathrm{Ext}^d_\infty(\mathcal{E}^\infty, \mathrm{St}^\infty_\mathfrak{p}(\Omega)) = 0$$

for all $d \geq 0$. But this follows directly from applying $\mathrm{Ext}_{G_\mathfrak{p}}(\cdot, \mathrm{St}^\infty_\mathfrak{p}(\Omega))$ to the short exact sequence (1.9) and the computations (1.7) and (1.8) of dimensions of smooth Ext-groups. □

Note that the approach we take here is the one of [22], Lemma 3.7.

2. Stabilizations

We explain the connection between overconvergent cohomology and the cohomology of $\mathfrak{p}$-arithmetic subgroups with values in duals of locally analytic principal series representations. We fix a prime $\mathfrak{p}$ of F, at which the quaternion algebra D is split, and an embedding

$$\iota_p \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p},$$

where p denotes the rational prime underlying $\mathfrak{p}$. We define E_p to be the completion of E with respect to the topology induced by ι_p.

2.1. **Smooth principal series.** We will give a representation-theoretic description of $\mathfrak{p}$-stabilizations as in [19], Section 2.2. Most of the basic results on smooth representations of PGL_2 over local fields that are used in this section can be found in [11], Section 4.5.

As mentioned before, we may identify $G_\mathfrak{p}$ with $\mathrm{PGL}_2(F_\mathfrak{p})$. Let B be the standard Borel subgroup of $G_\mathfrak{p}$ of upper triangular matrices.

Given a smooth Ω-representation τ of B its *smooth parabolic induction* is the space

$$i_B(\tau) = \{f \colon G_\mathfrak{p} \to \tau \text{ locally constant} \mid f(bg) = b.f(g)\ \forall b \in B,\ g \in G_\mathfrak{p}\}.$$

The group $G_\mathfrak{p}$ acts on $i_B(\tau)$ via right translation.

We identify the spaces of locally constant characters on B with those on $F^\times_\mathfrak{p}$ by mapping a locally constant character $\chi\colon F^\times_\mathfrak{p} \longrightarrow \Omega^\times$ to the character

$$B^\times \longrightarrow \Omega^\times, \quad \begin{pmatrix} b_1 & u \\ 0 & b_2 \end{pmatrix} \longmapsto \chi(b_1/b_2). \tag{2.1}$$

Definition 2.1. A $\mathfrak{p}$-stabilization (χ, ϑ) of π_Ω consists of a locally constant character $\chi\colon F^\times_\mathfrak{p} \longrightarrow \Omega^\times$ together with a non-zero $G_\mathfrak{p}$-equivariant homomorphism

$$\vartheta\colon i_B(\chi) \longrightarrow \pi_{\mathfrak{p},\Omega}.$$

Let (χ, ϑ) be a $\mathfrak{p}$-stabilization of π_Ω. Since $\pi_{\mathfrak{p},\Omega}$ is irreducible, ϑ is automatically surjective. Moreover, (χ, ϑ) induces a $\mathfrak{p}$-stabilization of $\pi_{\Omega'}$ for every extension Ω' of Ω.

By the classification of smooth irreducible representations of $G_\mathfrak{p}$ we know that $\pi_\mathfrak{p}$ is either supercuspidal or a quotient of a smooth parabolic induction as above (with $\Omega = \mathbb{C}$). In the first case, π_Ω admits no stabilization. In the later case, there always exists a finite extension Ω' of Ω such that $\pi_{\Omega'}$ admits a $\mathfrak{p}$-stabilization (χ, ϑ). Furthermore, the map ϑ is unique up to multiplication with a scalar.

The character χ is in general not unique: Suppose $\pi_{\Omega'}$ admits a $\mathfrak{p}$-stabilization (χ, ϑ). If ϑ is an isomorphism, then $\chi^2 \neq \mathbb{1}$ and $\chi^2 \neq |\cdot|_\mathfrak{p}^2$. Moreover, $i_B(\chi)$ is isomorphic to $i_B(\chi^{-1}|\cdot|_\mathfrak{p})$ but to no other principal series. Thus, as long as $\chi^2 \neq |\cdot|_\mathfrak{p}$ there are two essentially different stabilizations.

If ϑ is not an isomorphism, then $\chi^2 = \mathbb{1}$ and the kernel of ϑ is one-dimensional generated by the function $g \mapsto \chi(\det(g))$. Moreover, the sequence

$$0 \longrightarrow \ker(\vartheta) \longrightarrow i_B(\chi) \longrightarrow \pi_{\mathfrak{p},\Omega} \longrightarrow 0$$

is non-split. In particular, we have $\pi_{\mathfrak{p},\Omega} \cong \mathrm{St}_\mathfrak{p}^\infty(\Omega) \otimes \chi$ in this case.

From the discussion above and Proposition 1.4 we get the following:

Corollary 2.2. *Suppose π_Ω admits a $\mathfrak{p}$-stabilization (χ, ϑ). Then there is an isomorphism*

$$\mathrm{H}^q_\Omega(X^\mathfrak{p}_{K^\mathfrak{p}}, i_B(\chi), V_{\boldsymbol{k}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi} \cong (\pi_\Omega^{\mathfrak{p},\infty})^{K_\mathfrak{p}}$$

of $\mathbb{T}_{K^\mathfrak{p}}(\Omega)$-modules. In particular, it is an absolutely irreducible $\mathbb{T}_{K^\mathfrak{p}}(\Omega)$-module.

Remark 2.3. We will be mostly interested in the case that the representation $\pi_\mathfrak{p}$ admits an invariant vector under the Iwahori group

$$I_\mathfrak{p} = \{g \in \mathrm{PGL}_2(\mathcal{O}_\mathfrak{p}) \mid g \text{ is upper triangular mod } \mathfrak{p}\}.$$

It is well known that having an Iwahori fixed vector is equivalent to having a stabilization (χ, ϑ) with respect to an unramified character χ, i.e., $\chi(\mathcal{O}_\mathfrak{p}^*) = 1$.

2.2. **Non-critical stabilizations.** Composing an embedding $\sigma\colon F \hookrightarrow E \subseteq \overline{\mathbb{Q}}$ with ι_p induces a p-adic topology on F. We define $\Sigma_\mathfrak{p} \subseteq \Sigma$ to be the set of all embeddings inducing the topology coming from our chosen prime $\mathfrak{p}$ and put $\Sigma^\mathfrak{p} = \Sigma \setminus \Sigma_\mathfrak{p}$. We identify $\Sigma_\mathfrak{p} \subseteq \Sigma$ with the set of embeddings $F_\mathfrak{p} \hookrightarrow E_p$.

Suppose Ω is a finite extension of E_p, which we will do from now on. We may decompose

$$V_{\boldsymbol{k}}(\Omega) = V_{\boldsymbol{k}_\mathfrak{p}}(\Omega) \otimes_\Omega V_{\boldsymbol{k}^\mathfrak{p}}(\Omega)$$

with

$$V_{\boldsymbol{k}_\mathfrak{p}}(\Omega) = \bigotimes_{\sigma \in \Sigma_\mathfrak{p}} V_{k_\sigma}(\Omega) \qquad \text{and} \qquad V_{\boldsymbol{k}^\mathfrak{p}}(\Omega) = \bigotimes_{\sigma \in \Sigma^\mathfrak{p}} V_{k_\sigma}(\Omega).$$

The representation $V_{\boldsymbol{k}_\mathfrak{p}}(\Omega)$ of $G(F)$ extends to an algebraic representation of the group $G_\mathfrak{p}$.

We will assume for the reminder of this section that π_Ω admits a $\mathfrak{p}$-stabilization (χ, ϑ). We define the locally algebraic $G_\mathfrak{p}$-representation

$$i_B(\chi_{\boldsymbol{k}_\mathfrak{p}}) = i_B(\chi) \otimes_\Omega V_{\boldsymbol{k}_\mathfrak{p}}(\Omega).$$

Then, by (1.3) we have a canonical $\mathbb{T}^\mathfrak{p}_{K^\mathfrak{p}}(\Omega)$-equivariant isomorphism

$$\mathrm{H}^d_\Omega(X^\mathfrak{p}_{K^\mathfrak{p}}, i_B(\chi), V_{\boldsymbol{k}}(\Omega)^\vee) \xrightarrow{\cong} \mathrm{H}^d_\Omega(X^\mathfrak{p}_{K^\mathfrak{p}}, i_B(\chi_{\boldsymbol{k}_\mathfrak{p}}), V_{\boldsymbol{k}^\mathfrak{p}}(\Omega)^\vee).$$

Let A be an affinoid $\mathbb{Q}_p$-algebra. As in 2.1 we identify locally $\mathbb{Q}_p$-analytic characters from $F_\mathfrak{p}^\times$ to $A^\times$ with those from B to $A^\times$. Given a locally analytic representation τ of B its *locally analytic parabolic induction* is given by

$$\mathbb{I}_B(\tau) = \{f\colon G_\mathfrak{p} \to \tau \text{ locally analytic} \mid f(bg) = b.f(g)\ \forall b \in B,\ g \in G_\mathfrak{p}\}.$$

The group $G_\mathfrak{p}$ acts on it via right translation. Suppose that $A = \Omega$ and τ is finite-dimensional. In that case the locally analytic parabolic induction is a strongly admissible locally analytic representation of $G_\mathfrak{p}$. The case of one-dimensional representations is Proposition 1.21 of [17]. The proof works verbatim for finite-dimensional representations.

To any locally constant character $\chi\colon F_\mathfrak{p}^\times \to \Omega^\times$ we associate the locally analytic character $\chi_{\boldsymbol{k}_\mathfrak{p}}$ by

$$\chi_{\boldsymbol{k}_\mathfrak{p}}(x) = \chi(x) \prod_{\sigma\in\Sigma_\mathfrak{p}} \sigma(x)^{-k_\sigma/2}.$$

We may identify $i_B(\chi_{\boldsymbol{k}_\mathfrak{p}})$ with a subspace of $\mathbb{I}_B(\chi_{\boldsymbol{k}_\mathfrak{p}})$ via the map

$$\begin{aligned}\beta\colon i_B(\chi) \otimes \bigotimes_{\sigma\in\Sigma_\mathfrak{p}} V_{k_\sigma}(\Omega) &\longrightarrow \mathbb{I}_B(\chi_{\boldsymbol{k}_\mathfrak{p}})\\ (f_\infty, (f_\sigma)_{\sigma\in\Sigma_\mathfrak{p}}) &\longmapsto f_\infty \cdot \prod_{\sigma\in\Sigma_\mathfrak{p}} \psi_{f_\sigma}.\end{aligned}$$

See Section 1.1.1 for the definition of the functions ψ_{f_σ}.

Definition 2.4. A $\mathfrak{p}$-stabilization (χ, ϑ) of π_Ω is called non-critical if the canonical map

$$\beta^*\colon \mathrm{H}^d_{\Omega,\mathrm{cont}}(X^\mathfrak{p}_{K^\mathfrak{p}}, \mathbb{I}_B(\chi_{\boldsymbol{k}_\mathfrak{p}}), V_{\boldsymbol{k}^\mathfrak{p}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \longrightarrow \mathrm{H}^d_\Omega(X^\mathfrak{p}_{K^\mathfrak{p}}, i_B(\chi_{\boldsymbol{k}_\mathfrak{p}}), V_{\boldsymbol{k}^\mathfrak{p}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi}$$

is an isomorphism for all $d \geq 0$.

Note that the notion of non-criticality depends on the level $K^\mathfrak{p}$ away from $\mathfrak{p}$ and the set S. The notion gets stronger as smaller $K^\mathfrak{p}$ and as bigger S get.

2.2.1. *Locally algebraic and locally analytic Steinberg representation.* Assume for the moment that $\pi_\mathfrak{p}$ is the Steinberg representation. As mentioned above there is a unique $\mathfrak{p}$-stabilization $\vartheta\colon i_B(\mathbb{1}) \to \mathrm{St}^\infty_\mathfrak{p}(\Omega)$ which has a one dimensional kernel. We say π is non-critical at $\mathfrak{p}$ if this unique $\mathfrak{p}$-stabilization is non-critical.

We define the *locally algebraic Steinberg representation of weight* $\boldsymbol{k}_\mathfrak{p}$ via

$$\mathrm{St}^\infty_{\boldsymbol{k}_\mathfrak{p}}(\Omega) = \mathrm{St}^\infty_\mathfrak{p}(\Omega) \otimes_\Omega V_{\boldsymbol{k}_\mathfrak{p}}(\Omega)$$

and the *locally analytic Steinberg representation of weight* $\boldsymbol{k}_\mathfrak{p}$ as the quotient

$$\mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_\mathfrak{p}}(\Omega) = \mathbb{I}_B(\chi_{\boldsymbol{k}_\mathfrak{p}})/V_{\boldsymbol{k}_\mathfrak{p}}(\Omega).$$

Thus, we have a natural embedding

$$\kappa\colon\ \mathrm{St}^\infty_{\boldsymbol{k}_\mathfrak{p}}(\Omega) \hookrightarrow \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_\mathfrak{p}}(\Omega).$$

Unravelling the definitions we get the following statement.

Proposition 2.5. *Suppose that* $\pi_\mathfrak{p} \cong \mathrm{St}_\mathfrak{p}(\mathbb{C})$ *and* π *is non-critical. Then the canonical map*

$$\kappa^*\colon \mathrm{H}^d_{\Omega,\mathrm{cont}}(X^\mathfrak{p}_{K^\mathfrak{p}}, \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_\mathfrak{p}}(\Omega), V_{\boldsymbol{k}^\mathfrak{p}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \longrightarrow \mathrm{H}^d_\Omega(X^\mathfrak{p}_{K^\mathfrak{p}}, \mathrm{St}^\infty_{\boldsymbol{k}_\mathfrak{p}}(\Omega), V_{\boldsymbol{k}^\mathfrak{p}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi}$$

is an isomorphism for all $d \geq 0$.

Remark 2.6. On page 653 of [12] it is claimed that a property closely related to non-criticality always holds if the quaternion algebra D is totally definite. It is alluded to an Amice–Vélu and Vishik-type argument. But to the knowledge of the authors of this article the most general results of that type are in Section 7 of [10], which essentially only cover the case of non-critical slope.

In the following we are going to show that if the representation $\pi_\mathfrak{p}$ has an Iwahori-fixed vector the above definition of non-criticality is equivalent to the one given in terms of overconvergent cohomology that is used for example in [3] or [5]. In particular, the classicality theorem for overconvergent cohomology will give a numerical criterion for the non-criticality of a stabilization. In order to state this criterion later we will need the following definition.

Definition 2.7. Let (χ,ϑ) be a $\mathfrak{p}$-stabilization of π_Ω. The p-adic valuation of $\prod_{\sigma\in\Sigma_\mathfrak{p}}\sigma(\varpi_\mathfrak{p})^{\frac{k_\sigma}{2}}\chi(\omega_\mathfrak{p})$ is called the slope of (χ,ϑ). We say that (χ,ϑ) has non-critical slope if its slope is less than $\frac{1}{e_\mathfrak{p}}\min_{\sigma\in\Sigma_\mathfrak{p}}(k_\sigma+1)$.

2.3. **Overconvergent cohomology.** We give a quick overview over the basics of overconvergent cohomology.

2.3.1. *Locally analytic inductions.* Let

$$I_\mathfrak{p}^n=\{g\in \mathrm{PGL}_2(\mathcal{O}_\mathfrak{p})\mid g \text{ is upper triangular mod } \mathfrak{p}^n\}.$$

In particular, $I_\mathfrak{p}=I_\mathfrak{p}^1$ is the standard Iwahori subgroup.

Let A be an affinoid $\mathbb{Q}_p$-algebra. Let $\chi\colon B\cap I_\mathfrak{p}\to A^\times$ be a locally analytic character. This means that there exists a minimal integer $n_\chi\geq 1$ such that χ restricted to $B\cap I_\mathfrak{p}^{n_\chi}$ is analytic. For any integer $n\geq n_\chi$ define the $A[I_\mathfrak{p}]$-module $\mathcal{A}_\chi^n$ of functions $f\colon I_\mathfrak{p}\longrightarrow A$ such that

- f is analytic on any coset of $I_\mathfrak{p}/I_\mathfrak{p}^n$,
- $f(bk)=\chi(b)f(k)\quad \forall b\in B\cap I_\mathfrak{p},\ k\in I_\mathfrak{p}$

and put

$$\mathcal{A}_\chi=\bigcup_{n\geq n_\chi}\mathcal{A}_\chi^n.$$

The $A[I_\mathfrak{p}]$-module $\mathcal{A}_\chi$ is the locally analytic induction of χ to $I_\mathfrak{p}$. In the special case that $A=\Omega$ is a finite extension of E_p and $\chi=\mathbb{1}_{\boldsymbol{k}_\mathfrak{p}}$ we put

$$\mathcal{A}_{\boldsymbol{k}_\mathfrak{p}}=\mathcal{A}_{\mathbb{1}_{\boldsymbol{k}_\mathfrak{p}}}.$$

The Iwahori decomposition gives an isomorphism $I_\mathfrak{p}\cong(I_\mathfrak{p}\cap\overline{\mathbf{N}})\times B(\mathcal{O}_{F_\mathfrak{p}})$, where $\overline{\mathbf{N}}$ denotes the group of unipotent lower triangular matrices. Thus, restricting a function $f\in\mathcal{A}_\chi$ to $I_\mathfrak{p}\cap\overline{\mathbf{N}}$ induces an isomorphism between $\mathcal{A}_\chi$ and the space $\mathcal{A}(I_\mathfrak{p}\cap\overline{\mathbf{N}},A)$ of locally analytic A-valued functions on $I_\mathfrak{p}\cap\overline{\mathbf{N}}$. An analogous bijection holds between $\mathcal{A}_\chi^n$ and $\mathcal{A}_n(I_\mathfrak{p}\cap\overline{\mathbf{N}},A)$, the space of n-locally analytic functions on $I_\mathfrak{p}\cap\overline{\mathbf{N}}$.

2.3.2. *The $U_\mathfrak{p}$ operator.* Consider the compact induction $\text{c-ind}_{I_\mathfrak{p}}^{G_\mathfrak{p}}(\mathcal{A}_\chi^n)$. By Frobenius reciprocity, the ring $\mathrm{End}_{A[G_\mathfrak{p}]}(\text{c-ind}_{I_\mathfrak{p}}^{G_\mathfrak{p}}(\mathcal{A}_\chi^n))$ can be identified with the space of all functions $\Psi\colon G_\mathfrak{p}\longrightarrow \mathrm{End}_A(\mathcal{A}_\chi^n)$ such that

- Ψ is $I_\mathfrak{p}$-biequivariant, that is $\Psi(k_1gk_2)=k_1\Psi(g)k_2$ in $\mathrm{End}_A(\mathcal{A}_\chi^n)$, for all $k_1,k_2\in I_\mathfrak{p}$, $g\in G_\mathfrak{p}$, and
- for any element $f\in\mathcal{A}_\chi^n$, the function $G_\mathfrak{p}\to\mathcal{A}_\chi^n$, $g\mapsto\Psi(g)(f)$ is compactly supported.

Let $u_\mathfrak{p}:=\begin{pmatrix}\varpi_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}$. Consider the element $\varphi_{u_\mathfrak{p}}\in\mathrm{End}_A(\mathcal{A}_\chi^n)$ defined by

$$\varphi_{u_\mathfrak{p}}(f)(\overline{n})=f(u_\mathfrak{p}\overline{n}u_\mathfrak{p}^{-1})\qquad \text{for all } f\in\mathcal{A}_\chi^n,\ \overline{n}\in I_\mathfrak{p}\cap\overline{\mathbf{N}}.$$

By [27, Lemma 2.2], there exists a unique $I_\mathfrak{p}$-biequivariant function $\Psi_{u_\mathfrak{p}}\colon G_\mathfrak{p}\to\mathrm{End}_A(\mathcal{A}_\chi^n)$ such that $\mathrm{supp}(\Psi_{u_\mathfrak{p}})=I_\mathfrak{p}u_\mathfrak{p}^{-1}I_\mathfrak{p}$ and $\Psi_{u_\mathfrak{p}}(u_\mathfrak{p}^{-1})=\varphi_{u_\mathfrak{p}}$.

Abusing notation, we will simply denote by $u_\mathfrak{p}$ the $G_\mathfrak{p}$-equivariant endomorphism of $\text{c-ind}_{I_\mathfrak{p}}^{G_\mathfrak{p}}(\mathcal{A}_\chi^n)$ corresponding to $\Psi_{u_\mathfrak{p}}$.

For $d \geq 0$ we set

$$\mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\chi,\boldsymbol{k}^\mathfrak{p}}) = \mathrm{H}^d_{A,\mathrm{cont}}(X^\mathfrak{p}_{K^\mathfrak{p}}, \text{c-ind}_{I_\mathfrak{p}}^{G_\mathfrak{p}}(\mathcal{A}_\chi^n), A\otimes_\Omega V_{\boldsymbol{k}^\mathfrak{p}}(\Omega)^\vee).$$

For now, we are mostly interested in the following special case: $A=\Omega$ is a finite extension of E_p and $\chi = \mathbb{1}_{\boldsymbol{k}_\mathfrak{p}}$. In this situation we abbreviate

$$\mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\boldsymbol{k}}) = \mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\mathbb{1}_{\boldsymbol{k}_\mathfrak{p}},\boldsymbol{k}^\mathfrak{p}}).$$

Later we will also need the case that A is the coordinate ring of an affinoid subspace of the weight space (see Section 4.1).

Remark 2.8. In [3] overconvergent cohomology groups are introduced depending on a subset of the set of all primes of F lying above p. The spaces defined above correspond to the subset consisting only of the prime $\mathfrak{p}$.

The endomorphism $u_\mathfrak{p}$ induces an operator, that we denote by $U_\mathfrak{p}^\circ$, on cohomology:

$$U_\mathfrak{p}^\circ\colon H^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\chi,\boldsymbol{k}^\mathfrak{p}}) \longrightarrow H^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\chi,\boldsymbol{k}^\mathfrak{p}}).$$

Similar as before we may identify $V_{\boldsymbol{k}_\mathfrak{p}}(\Omega)$ with the space of (globally) algebraic vectors in $\mathcal{A}^n_{\boldsymbol{k}_\mathfrak{p}}$. It induces an embedding

$$\text{c-ind}_{I_\mathfrak{p}}^{G_\mathfrak{p}}(V_{\boldsymbol{k}_\mathfrak{p}}(\Omega)) \longrightarrow \text{c-ind}_{I_\mathfrak{p}}^{G_\mathfrak{p}}(\mathcal{A}^n_{\boldsymbol{k}_\mathfrak{p}})$$

and the subspace $\text{c-ind}_{I_\mathfrak{p}}^{G_\mathfrak{p}}(V_{\boldsymbol{k}_\mathfrak{p}}(\Omega))$ is clearly invariant under the action of $u_\mathfrak{p}$.

Thus by invoking (1.5) we get a $\mathbb{T}^\mathfrak{p}_{K^\mathfrak{p}}(\Omega)$-equivariant map

$$\mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\boldsymbol{k}}) \longrightarrow \mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, V^\vee_{\boldsymbol{k}}) \tag{2.2}$$

in cohomology. We denote the natural operator on the right hand side induced by $u_\mathfrak{p}$ by $U_\mathfrak{p}$. The map (2.2) intertwines the action of the Hecke operator $U_\mathfrak{p}^\circ$ on $H^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\boldsymbol{k}})$ with the action of $\prod_{\sigma\in\Sigma_\mathfrak{p}}\sigma(\varpi_\mathfrak{p})^{\frac{k_\sigma}{2}}U_\mathfrak{p}$ on $H^d_c(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, V^\vee_{\boldsymbol{k}})$. This follows from a simple analysis of the change of action of $u_\mathfrak{p}$ under the isomorphism

$$\text{c-ind}_{I_\mathfrak{p}}^{G_\mathfrak{p}} V_{\boldsymbol{k}_\mathfrak{p}} \cong V_{\boldsymbol{k}_\mathfrak{p}} \otimes_\Omega \text{c-ind}_{I_\mathfrak{p}}^{G_\mathfrak{p}}\Omega$$

given by (1.4). We define the Hecke operator $U_\mathfrak{p}^\circ$ on the right hand side of (2.2) by

$$U_\mathfrak{p}^\circ = \prod_{\sigma\in\Sigma_\mathfrak{p}}\sigma(\varpi_\mathfrak{p})^{\frac{k_\sigma}{2}}U_\mathfrak{p}$$

and similarly we define an action of U_p on the left hand side of (2.2).

2.3.3. *Slope decompositions and classicality.* We give a reminder on slope decompositions. As before, A denotes a affinoid $\mathbb{Q}_p$-algebra. Let M be an A-module equipped with an A-linear endomorphism $u\colon M\to M$. Fix a rational number $h\geq 0$. A polynomial $Q\in A[x]$ is multiplicative of slope $\leq h$ if

- the leading coefficient of Q is a unit in A and
- every edge of the Newton polygon of Q has slope $\leq h$.

We put $Q^*(x) = x^{\deg Q}Q(1/x)$. An element $m\in M$ is said to be of slope $\leq h$ if there is a multiplicative polynomial $Q\in A[x]$ of slope $\leq h$ such that $Q^*(u)m = 0$. Let $M^{\leq h}\subseteq M$ be the submodule of elements of M of slope $\leq h$.

Definition 2.9. A slope $\leq h$ decomposition of M is an $A[u]$-module isomorphism

$$M\cong M^{\leq h}\oplus M^{>h}$$

such that

- $M^{\leq h}$ is a finitely generated A-module and
- $Q^*(u)$ acts invertibly on $M^{>h}$ for every multiplicative polynomial $Q \in A[x]$ of slope $\leq h$.

Note that if $A = \Omega$ is a finite extension of E_p and M is finite-dimensional, then a slope $\leq h$ decomposition always exists. If M admits a slope $\leq h$ decomposition for all $h \geq 0$, we put

$$M^{<\infty} = \bigcup_{h\geq 0} M^{\leq h}.$$

The most remarkable result about slope decomposition is the following theorem which was first proved by Ash and Stevens [1] in special cases over $\mathbb{Q}$ and then generalized by Urban [37] and Hansen [24] to more general settings. See also [4] for a detailed treatment if the case of PGL_2 over arbitrary number fields. In these results one always considers all primes above $\mathfrak{p}$ simultaneously. The modifications necessary to allow subsets of all primes above $\mathfrak{p}$ are explained in the proof of [3], Theorem 2.7.

Theorem 2.10. *For every $d \geq 0$ and every $h \geq 0$ the cohomology groups*

$$H^d(X_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}})$$

admit a slope $\leq h$ decomposition with respect to the Hecke operator $U^{\circ}_{\mathfrak{p}}$.

If $h < \frac{1}{e_{\mathfrak{p}}} \min_{\sigma\in\Sigma_{\mathfrak{p}}}(k_\sigma + 1)$, where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$, then for all $d \geq 0$ the map (2.2) *induces the following $\mathbb{T}^{\mathfrak{p}}_{K^{\mathfrak{p}}}$-equivariant isomorphism*

$$H^d(X_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, \mathcal{D}^n_{\boldsymbol{k}})^{\epsilon,\leq h} \xrightarrow{\sim} H^d(X_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, V^{\vee}_{\boldsymbol{k}})^{\epsilon,\leq h}.$$

Here the slope decomposition is taken with respect to $U^{\circ}_{\mathfrak{p}}$ on both sides.

2.4. **Overconvergent cohomology and non-critical stabilization.** Let A be an affinoid algebra and $\chi\colon F^{\times}_{\mathfrak{p}} \longrightarrow A^{\times}$ be a locally analytic character. Denote by χ_0 its restriction to $\mathcal{O}^{\times}_{F_{\mathfrak{p}}}$. An element $f \in \mathcal{A}^n_{\chi_0}$ can uniquely extended to a function on $BI_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ by putting $f(bk) = \chi(b)f(k)$. Since $BI_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ is open, extension by zero yields an $I_{\mathfrak{p}}$-equivariant A-linear injection

$$\mathcal{A}^n_{\chi_0} \hookrightarrow \mathbb{I}_B(\chi)|_{I_{\mathfrak{p}}}.$$

By Frobenius reciprocity, it induces a unique $G_{\mathfrak{p}}$-equivariant A-linear morphism

$$\mathrm{aug}_\chi\colon \text{c-ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\mathcal{A}^n_{\chi_0}) \longrightarrow \mathbb{I}_B(\chi). \tag{2.3}$$

The following theorem is due to Kohlhaase and Schraen.

Theorem 2.11. *For $n \geq n_{\chi_0}$, the short sequence*

$$0 \longrightarrow \text{c-ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\mathcal{A}^n_{\chi_0}) \xrightarrow{u_{\mathfrak{p}}-\chi(\varpi_{\mathfrak{p}})} \text{c-ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\mathcal{A}^n_{\chi_0}) \xrightarrow{\mathrm{aug}_\chi} \mathbb{I}_B(\chi) \longrightarrow 0 \tag{2.4}$$

is exact, where aug_χ is the map obtained by composing (2.3) *with the natural map* $\text{c-ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\mathcal{A}^n_{\chi_0}) \to \text{c-ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\mathcal{A}_{\chi_0})$.

Proof. See [27, Proposition 2.4 and Theorem 2.5]. □

Let $\chi\colon F^{\times}_{\mathfrak{p}} \to \Omega^{\times}$ be a smooth character. Similarly as above, the sequence

$$0 \longrightarrow \text{c-ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\Omega) \xrightarrow{u_{\mathfrak{p}}-\chi(\varpi_{\mathfrak{p}})} \text{c-ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\Omega) \xrightarrow{\mathrm{aug}_\chi} i_B(\chi) \longrightarrow 0$$

is exact. This can be deduced from Borel's theorem that $\text{c-ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(\Omega)$ is a flat module over the Iwahori-Hecke algebra (see the end of Section 3.1 of [23]). Tensoring the above exact sequence with $V_{\boldsymbol{k}_{\mathfrak{p}}}$ and using (1.4) we get a short exact sequence

$$0 \longrightarrow \text{c-ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(V_{\boldsymbol{k}_{\mathfrak{p}}}) \xrightarrow{u_{\mathfrak{p}}-\chi_{\boldsymbol{k}_{\mathfrak{p}}}(\varpi_{\mathfrak{p}})} \text{c-ind}^{G_{\mathfrak{p}}}_{I_{\mathfrak{p}}}(V_{\boldsymbol{k}_{\mathfrak{p}}}) \xrightarrow{\mathrm{aug}_{\chi_{\boldsymbol{k}_{\mathfrak{p}}}}} i_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}) \longrightarrow 0. \tag{2.5}$$

Given a $\mathfrak{p}$-stabilization (χ,ϑ) of π_Ω we define

$$\mathfrak{m}^S_{\pi,(\chi,\vartheta)} \subseteq \mathbb{T}^S_K(\Omega)[U_\mathfrak{p}]$$

to be the maximal ideal generated by $\mathfrak{m}^S_\pi$ and $U_\mathfrak{p} - \chi(\varpi_\mathfrak{p})$. In accordance with [3], Definition 2.12, and [5], Definition 1.5.1, we make the following definition:

Definition 2.12. The maximal ideal $\mathfrak{m}^S_{\pi,(\chi,\vartheta)} \subseteq \mathbb{T}^S_K(\Omega)[U_\mathfrak{p}]$ is non-critical if the map

$$\mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\boldsymbol{k}})^{<\infty}_{\mathfrak{m}^S_{\pi,(\chi,\vartheta)}} \longrightarrow \mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, V^\vee_{\boldsymbol{k}})_{\mathfrak{m}^S_{\pi,(\chi,\vartheta)}}$$

induced by (2.2) is isomorphisms for all $d \geq 0$.

Proposition 2.13. *Suppose that $\pi_\mathfrak{p}$ has an Iwahori fixed vector and let (χ,ϑ) be a $\mathfrak{p}$-stabilization of π_Ω. Then the following are equivalent:*

(i) (χ,ϑ) is non-critical
(ii) $\mathfrak{m}^S_{\pi,(\chi,\vartheta)}$ is non-critical.

Proof. The long exact sequences induced by (2.4) and (2.5) yield the following diagram with exact columns:

$$\begin{array}{ccc}
\mathrm{H}^d_{\Omega,\mathrm{cont}}(X^\mathfrak{p}_{K^\mathfrak{p}}, \mathbb{I}_B(\chi_{\boldsymbol{k}_\mathfrak{p}}), V_{\boldsymbol{k}^\mathfrak{p}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} & \xrightarrow{\ \beta^*\ } & \mathrm{H}^d_{\Omega}(X^\mathfrak{p}_{K^\mathfrak{p}}, i_B(\chi_{\boldsymbol{k}_\mathfrak{p}}), V_{\boldsymbol{k}^\mathfrak{p}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \\
\Big\downarrow{\scriptstyle \mathrm{aug}^*_{\chi_{\boldsymbol{k}_\mathfrak{p}}}} & & \Big\downarrow{\scriptstyle \mathrm{aug}^*_{\chi_{\boldsymbol{k}_\mathfrak{p}}}} \\
\mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\boldsymbol{k}})_{\mathfrak{m}^S_\pi} & \xrightarrow{\ (2.2)\ } & \mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \\
\Big\downarrow{\scriptstyle U^\circ_\mathfrak{p} - \chi_{\boldsymbol{k}_\mathfrak{p}}(\varpi_\mathfrak{p})} & & \Big\downarrow{\scriptstyle U^\circ_\mathfrak{p} - \chi_{\boldsymbol{k}_\mathfrak{p}}(\varpi_\mathfrak{p})} \\
\mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\boldsymbol{k}})_{\mathfrak{m}^S_\pi} & \xrightarrow{\ (2.2)\ } & \mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, V_{\boldsymbol{k}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} \\
\Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial} \\
\mathrm{H}^{d+1}_{\Omega,\mathrm{cont}}(X^\mathfrak{p}_{K^\mathfrak{p}}, \mathbb{I}_B(\chi_{\boldsymbol{k}_\mathfrak{p}}), V_{\boldsymbol{k}^\mathfrak{p}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi} & \xrightarrow{\ \beta^*\ } & \mathrm{H}^{d+1}_{\Omega}(X^\mathfrak{p}_{K^\mathfrak{p}}, i_B(\chi_{\boldsymbol{k}_\mathfrak{p}}), V_{\boldsymbol{k}^\mathfrak{p}}(\Omega)^\vee)_{\mathfrak{m}^S_\pi}
\end{array}$$

From the existence of slope decompositions (see Theorem 2.10) we may replace $\mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\boldsymbol{k}})_{\mathfrak{m}^S_\pi}$ by $\mathrm{H}^d(X_{K^\mathfrak{p}\times I_\mathfrak{p}}, \mathcal{D}^n_{\boldsymbol{k}})_{\mathfrak{m}^S_{\pi,(\chi,\vartheta)}}$ in diagram above (and similarly for cohomology with coefficients in $V_{\boldsymbol{k}}$). The claim then follows by induction on d. □

Applying the second part of Theorem 2.10 we get the following:

Corollary 2.14. *Suppose that $\pi_\mathfrak{p}$ has an Iwahori-fixed vector. If a $\mathfrak{p}$-stabilization (χ,ϑ) of π_Ω has non-critical slope, then it is non-critical.*

Suppose $\pi_\mathfrak{p} = \mathrm{St}_\mathfrak{p}(\mathbb{C})$. Then the corollary above shows that π is non-critical at $\mathfrak{p}$ if

(i) $k_\sigma = 0$ for all $\sigma \in \Sigma_\mathfrak{p}$ or
(ii) $F_\mathfrak{p} = \mathbb{Q}_p$ or
(iii) $[F_\mathfrak{p} : \mathbb{Q}_p] = 2$ and $k_{\sigma_1} = k_{\sigma_2}$ where $\Sigma_\mathfrak{p} = \{\sigma_1, \sigma_2\}$.

In particular, this holds if $F = \mathbb{Q}$ or if F is imaginary quadratic by (1.1).

3. Automorphic L-invariants

The main aim of this section is to define automorphic $\mathcal{L}$-invariants for the representation π under the assumption that the local component of π at a prime $\mathfrak{p}$ is Steinberg.

3.1. **Extensions of locally analytic Steinberg representations.** The following construction of extensions is due to Breuil (see [8], Section 2.1). Let $\lambda\colon F_{\mathfrak{p}}^{\times} \to \Omega$ be a continuous homomorphism. Note that λ is automatically locally $\mathbb{Q}_p$-analytic. We define τ_λ to be the two dimensional Ω-representation of B given by

$$\begin{pmatrix} a & u \\ 0 & d \end{pmatrix} \longmapsto \begin{pmatrix} 1 & \lambda(a/d) \\ 0 & 1 \end{pmatrix}$$

and put $\tau_{\lambda,\boldsymbol{k}_{\mathfrak{p}}} = \tau_\lambda \otimes \chi_{\boldsymbol{k}_{\mathfrak{p}}}$. The short exact sequence

$$0 \longrightarrow \chi_{\boldsymbol{k}_{\mathfrak{p}}} \longrightarrow \tau_{\lambda,\boldsymbol{k}_{\mathfrak{p}}} \longrightarrow \chi_{\boldsymbol{k}_{\mathfrak{p}}} \longrightarrow 0$$

induces the short exact sequence

$$0 \longrightarrow \mathbb{I}_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}) \longrightarrow \mathbb{I}_B(\tau_{\lambda,\boldsymbol{k}_{\mathfrak{p}}}) \longrightarrow \mathbb{I}_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}) \longrightarrow 0$$

of locally analytic representations. Pullback via $V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \hookrightarrow \mathbb{I}_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}})$ and pushforward along $\mathbb{I}_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}) \twoheadrightarrow \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega)$ yields the exact sequence

$$0 \longrightarrow \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \longrightarrow \mathcal{E}_{\lambda,\boldsymbol{k}_{\mathfrak{p}}} \longrightarrow V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \longrightarrow 0. \tag{3.1}$$

Remark 3.1. Given two locally $\mathbb{Q}_p$-analytic Ω-representations W_1 and W_2 we denote by $\mathrm{Ext}^1_{\mathrm{an}}(W_1, W_2)$ the space of locally $\mathbb{Q}_p$-analytic extensions of W_2 by W_1. The map

$$\mathrm{Hom}_{\mathrm{cont}}(F_{\mathfrak{p}}^{\times}, \Omega) \longrightarrow \mathrm{Ext}^1_{\mathrm{an}}(V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega), \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega)),\ \lambda \longmapsto \mathcal{E}_{\lambda,\boldsymbol{k}_{\mathfrak{p}}}$$

is an isomorphism. In the case $F_{\mathfrak{p}} = \mathbb{Q}_p$ this is due to Breuil. In fact, an analogous statement is true for more general split reductive groups (see [16], Theorem 1, and [22], Theorem 2.15).

We put

$$\mathcal{E}^{\infty}_{\boldsymbol{k}_{\mathfrak{p}}} = \mathcal{E}^{\infty} \otimes_{\Omega} V_{\boldsymbol{k}_{\mathfrak{p}}}$$

where $\mathcal{E}^{\infty}$ is the smooth extension of (1.9). By definition we have an exact sequence

$$0 \longrightarrow \mathrm{St}^{\infty}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \longrightarrow \mathcal{E}^{\infty}_{\boldsymbol{k}_{\mathfrak{p}}} \longrightarrow V_{\boldsymbol{k}_{\mathfrak{p}}} \longrightarrow 0.$$

It is easy to see that the extension $\mathcal{E}^{\infty}_{\boldsymbol{k}_{\mathfrak{p}}}$ is mapped to a multiple of $\mathcal{E}_{\mathrm{ord}_{\mathfrak{p}},\boldsymbol{k}_{\mathfrak{p}}}$ under the map $\mathrm{Ext}^1_{\mathrm{an}}(V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega), \mathrm{St}^{\infty}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega)) \to \mathrm{Ext}^1_{\mathrm{an}}(V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega), \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega))$.

3.2. **Definition of the L-invariant.** We assume for the rest of this article that $\pi_{\mathfrak{p}}$ is the Steinberg representation.

For a continuous homomorphism $\lambda\colon F_{\mathfrak{p}}^{\times} \to \Omega$ we let

$$c^{\epsilon}_{\lambda}\colon \mathrm{H}^q_{\Omega,\mathrm{cont}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}^S_{\pi}} \longrightarrow \mathrm{H}^{q+1}_{\Omega}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}^S_{\pi}}$$

be the boundary map induced by the dual of the short exact sequence (3.1). This map is clearly a $\mathbb{T}^{\mathfrak{p}}_{K^{\mathfrak{p}}}(\Omega)$-module homomorphism.

Definition 3.2. The automorphic $\mathcal{L}$-invariant of π at $\mathfrak{p}$

$$\mathcal{L}_{\mathfrak{p}}(\pi)^{\epsilon} \subseteq \mathrm{Hom}_{\mathrm{cont}}(F_{\mathfrak{p}}^{\times}, \Omega)$$

is the kernel of the map $\lambda \mapsto c^{\epsilon}_{\lambda}$.

Proposition 3.3. *Assume that π is non-critical at $\mathfrak{p}$. Then the codimension of the $\mathcal{L}$-invariant $\mathcal{L}_{\mathfrak{p}}(\pi)^{\epsilon} \subseteq \mathrm{Hom}_{\mathrm{cont}}(F_{\mathfrak{p}}^{\times}, \Omega)$ is equal to one. Moreover, it does not contain the space of locally constant characters.*

Proof. Combining Proposition 2.5 and Lemma 1.8 we see that the space of $\mathbb{T}_{K^{\mathfrak{p}}}(\Omega)$-linear homomorphisms between the two modules $\mathrm{H}^q_{\Omega,\mathrm{cont}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}^S_{\pi}}$ and $\mathrm{H}^{q+1}_{\Omega}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}^S_{\pi}}$ is one-dimensional. Thus, the $\mathcal{L}$-invariant is of codimension at most one. By the remark at the end of Section 3.1 we have

$c^{\epsilon}_{\mathrm{ord}_{\mathfrak{p}}} = c^{\epsilon}_{\infty} \circ \kappa^{*}$. By Lemma 1.8 the homomorphism c^{ϵ}_{∞} is an isomorphism, while κ^{*} an isomorphism by Proposition 2.5. □

Remark 3.4. As in [20] one could also define automorphic $\mathcal{L}$-invariants for higher degree cohomology groups, for which its π-isotypic component does not vanish. As these $\mathcal{L}$-invariants neither seem to show up in exceptional zero formulas nor are they used to define (plectic) Darmon cycles we will not consider them here.

4. P-adic families

For this section we assume that F is totally real that $\pi_{\mathfrak{p}}(\mathbb{C})$ is the Steinberg representation and π is non-critical at $\mathfrak{p}$. We give a formula for the automorphic $\mathcal{L}$-invariant in terms of derivatives of $U_{\mathfrak{p}}$-eigenvalues of p-adic families passing through π. Comparing with the corresponding formula for the Fontaine–Mazur $\mathcal{L}$-invariant of the corresponding Galois representation we deduce that automorphic and Fontaine–Mazur $\mathcal{L}$-invariants agree.

4.1. **The weight space.** Let Ω be a finite extension of E_p. Define the (partial) weight space $\mathcal{W}_{\mathfrak{p}}$ to be the rigid analytic space over Ω associated to the completed group algebra $\mathcal{O}_{\Omega}[\![\mathcal{O}_{\mathfrak{p}}^{\times}]\!]$. There is a universal character

$$\kappa^{\mathrm{un}}\colon \mathcal{O}_{\mathfrak{p}}^{\times} \longrightarrow (\mathcal{O}_{\Omega}[\![\mathcal{O}_{\mathfrak{p}}^{\times}]\!])^{\times}.$$

Let $\mathcal{U} \subseteq \mathcal{W}_{\mathfrak{p}}$ be an affinoid and $\mathcal{O}(\mathcal{U})$ be the ring of its rigid analytic functions. We will denote by $\kappa^{\mathrm{un}}_{\mathcal{U}}\colon \mathcal{O}_{\mathfrak{p}}^{\times} \to \mathcal{O}(\mathcal{U})^{\times}$ the restriction of the universal character to $\mathcal{U}$. For an affinoid $\mathcal{U} \subseteq \mathcal{W}_{\mathfrak{p}}$ and a locally analytic character $\chi\colon B \cap I_{\mathfrak{p}} \to \mathcal{O}(\mathcal{U})^{\times}$ recall from section 2.3 the $\mathcal{O}(\mathcal{U})[I_{\mathfrak{p}}]$-module $\mathcal{A}^{n}_{\chi}$ defined as the locally n-analytic induction of χ to $I_{\mathfrak{p}}$, and the cohomology groups $\mathrm{H}^{d}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^{n}_{\chi,\boldsymbol{k}^{\mathfrak{p}}})$. If χ is the universal character $\kappa^{\mathrm{un}}_{\mathcal{U}}$, we simply write $\mathrm{H}^{d}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^{n}_{\mathcal{U},\boldsymbol{k}^{\mathfrak{p}}})$ in place of $\mathrm{H}^{d}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^{n}_{\kappa^{\mathrm{un}}_{\mathcal{U}},\boldsymbol{k}^{\mathfrak{p}}})$.

4.2. **Etaleness at $\mathfrak{m}_{\pi}$.** Let $\mathcal{U}$ be an admissible open affinoid in $\mathcal{W}_{\mathfrak{p}}$ containing $\boldsymbol{k}_{\mathfrak{p}}$ and let $\mathcal{O}(\mathcal{U})_{\boldsymbol{k}_p}$ be the rigid localization of $\mathcal{O}(\mathcal{U})$ at $\boldsymbol{k}_{\mathfrak{p}} \in \mathcal{U}$. It is the local ring defined as

$$\mathcal{O}(\mathcal{U})_{\boldsymbol{k}_{\mathfrak{p}}} = \varinjlim_{\boldsymbol{k}_{\mathfrak{p}} \in \mathcal{U}' \subset \mathcal{U}} \mathcal{O}(\mathcal{U}')$$

where the limit is taken over all admissible open sub-affinoid $\mathcal{U}'_{\mathfrak{p}}$ in $\mathcal{U}$ containing $\boldsymbol{k}_{\mathfrak{p}}$. Thus, it contains the algebraic localization of $\mathcal{O}(\mathcal{U})$ at the maximal ideal $\mathfrak{m}_{\boldsymbol{k}_{\mathfrak{p}}}$ of $\boldsymbol{k}_{\mathfrak{p}}$.

Theorem 4.1. *Up to shrinking $\mathcal{U}$ to a small enough open affinoid containing $\boldsymbol{k}_{\mathfrak{p}}$ the following holds:*

$$\mathrm{H}^{d}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^{n}_{\mathcal{U},\boldsymbol{k}^{\mathfrak{p}}})^{\epsilon}_{(\mathfrak{m}^{S}_{\pi}, U_{\mathfrak{p}}-1)} = 0 \quad \textit{for every } d \neq q,$$

for $d = q$ it is a free $\mathcal{O}(\mathcal{U})_{\boldsymbol{k}_{\mathfrak{p}}}$-module of finite rank and the map of $\mathcal{O}(\mathcal{U})_{\boldsymbol{k}_{\mathfrak{p}}}$-modules obtained by localizing the composition of (2.2) *with the map induced by reduction to $\mathcal{D}^{n}_{\boldsymbol{k}}$ induces an isomorphism*

$$\mathrm{H}^{q}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, \mathcal{D}^{n}_{\mathcal{U},\boldsymbol{k}^{\mathfrak{p}}})^{\epsilon}_{(\mathfrak{m}^{S}_{\pi}, U_{\mathfrak{p}}-1)} \otimes_{\mathcal{O}(\mathcal{U})_{\boldsymbol{k}_{\mathfrak{p}}}} \mathcal{O}(\mathcal{U})_{\boldsymbol{k}_{\mathfrak{p}}}/\mathfrak{m}_{\boldsymbol{k}_{\mathfrak{p}}} \xrightarrow{\cong} \mathrm{H}^{q}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, V^{\vee}_{\boldsymbol{k}})^{\epsilon}_{(\mathfrak{m}^{S}_{\pi}, U_{\mathfrak{p}}-1)}.$$

Moreover, the operator $U^{\circ}_{\mathfrak{p}}$ acts on it via a scalar $\alpha_{\mathfrak{p}} \in \mathcal{O}(\mathcal{U})^{\times}_{\boldsymbol{k}_{\mathfrak{p}}}$.

Proof. Recall that by Theorem 1.1 we have

$$\mathrm{H}^{d}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^{\vee})_{(\mathfrak{m}^{S}_{\pi}, U_{\mathfrak{p}}-1)} = 0 \quad \text{for all } d \neq q.$$

The first claims follow using the same arguments as in the proof of [3, Theorem 2.14]. The statement about the operator $U^{\circ}_{\mathfrak{p}}$ can be deduced from the fact that

$$\mathrm{H}^{q}(X_{K^{\mathfrak{p}} \times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^{\vee})^{\epsilon}_{(\mathfrak{m}^{S}_{\pi}, U_{\mathfrak{p}}-1)}$$

is an absolutely irreducible $\mathbb{T}_{K^{\mathfrak{p}}}(\Omega)$-module. □

4.3. **Infinitesimal deformations and $\mathcal{L}$-invariants.** Let $\Omega[\varepsilon] := \Omega[X]/(X^2)$ be the Ω-algebra of dual numbers over Ω and $\pi\colon \Omega[\varepsilon] \to \Omega$ be the the natural surjection sending ε to 0. It should be thought as the space of tangent vectors to a fixed Ω-scheme at a fixed point in the following sense. If $X = \operatorname{Spec}(A)$ is an affine Ω-scheme and $x\colon A \to A/\mathfrak{m}_x = \Omega$ is a Ω-valued point, then the space of morphisms $\boldsymbol{v}_x\colon A \to \Omega[\varepsilon]$ such that $\pi \circ \boldsymbol{v}_x = x$ is identified with the tangent space of X at x.

Let $\mathcal{U}$ be an admissible open affinoid containing $\boldsymbol{k}$ and $\chi\colon B^\times \to \mathcal{O}(\mathcal{U})^\times$ a locally analytic character, that we identify with an element of $\operatorname{Hom}(F_{\mathfrak{p}}^\times, \mathcal{O}(\mathcal{U})^\times)$ as in section 2.1.

Let $\boldsymbol{v}\colon \mathcal{O}(\mathcal{U}) \to \Omega[\varepsilon]$ be an element of the tangent space of $\mathcal{U}$ at $\boldsymbol{k}$. Then the pullback $\chi_{\boldsymbol{v}} = \boldsymbol{v} \circ \chi \in \operatorname{Hom}(F_{\mathfrak{p}}^\times, \Omega[\varepsilon])$ of χ along $\boldsymbol{v}$ can be written in a unique way as

$$\chi_{\boldsymbol{v}} = \overline{\chi}(1 + \partial_{\boldsymbol{v}}(\chi)\varepsilon), \tag{4.1}$$

where $\overline{\chi}\colon F_{\mathfrak{p}}^\times \to (\mathcal{O}(\mathcal{U})/\mathfrak{m}_{\boldsymbol{k}})^\times = \Omega^\times$ denotes the reduction of χ modulo $\mathfrak{m}_{\boldsymbol{k}}$, and $\partial_{\boldsymbol{v}}(\chi)$ is a homomorphism $F_{\mathfrak{p}}^\times \to \Omega$.

Now, assume that $\chi\colon B^\times \to \mathcal{O}(\mathcal{U})^\times$ is a locally analytic character such that $\chi \pmod{\mathfrak{m}_{\boldsymbol{k}}} = \mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}$. For an element $\boldsymbol{v}$ in the tangent space of $\mathcal{U}$ at $\boldsymbol{k}$, write $\chi_{\boldsymbol{v}} = \boldsymbol{v} \circ \chi$ as above $\chi_{\boldsymbol{v}} = \mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}(1 + \partial_{\boldsymbol{v}}(\chi)\varepsilon)$. Consider the map induced in cohomology by the reduction of χ modulo $\mathfrak{m}_{\boldsymbol{k}}$:

$$\operatorname{red}_\chi\colon \operatorname{H}^q_{\mathcal{O}(\mathcal{U}),\mathrm{cont}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\chi), V^\vee_{\boldsymbol{k}^{\mathfrak{p}}})^\epsilon_{\mathfrak{m}^S_\pi} \longrightarrow \operatorname{H}^q_{\Omega,\mathrm{cont}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}), V^\vee_{\boldsymbol{k}^{\mathfrak{p}}})^\epsilon_{\mathfrak{m}^S_\pi}.$$

Then, the following holds.

Proposition 4.2. *If* red_χ *is surjective, then* $\partial_{\boldsymbol{v}}(\chi)$ *belongs to* $\mathcal{L}_{\mathfrak{p}}(\pi)^\epsilon$ *for every element* $\boldsymbol{v}$ *of the tangent space of* $\mathcal{U}$ *at* $\boldsymbol{k}$.

Proof. The locally analytic character $\chi_{\boldsymbol{v}}$ of B over $\Omega[\varepsilon]$ can be seen as a two-dimensional representation $\tau_{\chi_{\boldsymbol{v}}}$ of B over Ω. It is in fact the representation that we denoted by $\tau_{\partial_{\boldsymbol{v}}(\chi),\boldsymbol{k}_{\mathfrak{p}}}$ in section 3.1. From the discussion in 3.1, there is a commutative diagram

$$\begin{array}{ccc}
\operatorname{H}^q(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi} & \xrightarrow{\hat{c}^\epsilon_{\partial_{\boldsymbol{v}}(\chi)}} & \operatorname{H}^{q+1}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi} \\
\uparrow & & \downarrow \\
\operatorname{H}^q(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi} & \xrightarrow{c^\epsilon_{\partial_{\boldsymbol{v}}(\chi)}} & \operatorname{H}^{q+1}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi}
\end{array}$$

where $\hat{c}^\epsilon_{\partial_{\boldsymbol{v}}(\chi)}$ and $c^\epsilon_{\partial_{\boldsymbol{v}}(\chi)}$ are the boundary maps induced by the dual of the short exact sequences

$$0 \to \mathbb{I}_B(\mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}) \to \mathbb{I}_B(\tau_{\chi_{\boldsymbol{v}}}) \to \mathbb{I}_B(\mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}) \to 0, \tag{4.2}$$

and

$$0 \to \mathrm{St}^{\mathrm{an}}_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \to \mathcal{E}_{\partial_{\boldsymbol{v}}(\chi),\boldsymbol{k}_{\mathfrak{p}}} \to V_{\boldsymbol{k}_{\mathfrak{p}}}(\Omega) \to 0,$$

respectively. It is sufficient to prove that $\hat{c}^\epsilon_{\partial_{\boldsymbol{v}}(\chi)}$ is the zero map. This follows from the fact that the map

$$\operatorname{H}^q(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\tau_{\chi_{\boldsymbol{v}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi} \to \operatorname{H}^q(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^\vee)^\epsilon_{\mathfrak{m}^S_\pi}$$

induced in cohomology by the dual of (4.2) is surjective because it is induced by the reduction $\chi_{\boldsymbol{v}} \to \mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}$ of $\chi_{\boldsymbol{v}}$ modulo ε, which is the same as the reduction $\chi \to \mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}$ modulo $\mathfrak{m}_{\boldsymbol{k}}$ for every $\boldsymbol{v}$ by definition of tangent space. □

Recall that we denoted by $\alpha_{\mathfrak{p}} \in \mathcal{O}(\mathcal{U})^{\times}_{\boldsymbol{k}_{\mathfrak{p}}}$ the eigenvalue of the Hecke operator $U_{\mathfrak{p}}^{\circ}$ acting on $\mathrm{H}^q(X_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, \mathcal{D}^n_{\mathcal{U},\boldsymbol{k}_{\mathfrak{p}}})^{\epsilon}_{(\mathfrak{m}_{\pi}^{S},U_{\mathfrak{p}}-1)}$. Up to shrinking $\mathcal{U}$ we can assume that $\alpha_{\mathfrak{p}} \in \mathcal{O}(\mathcal{U})^{\times}$.

Let $\chi_{\alpha_{\mathfrak{p}}} \colon B^{\times} \to \mathcal{O}(\mathcal{U})^{\times}$ be the character defined by

$$\chi_{\alpha_{\mathfrak{p}}}|_{B\cap I_{\mathfrak{p}}} = \kappa_{\mathcal{U}}^{\mathrm{un}}|_{\mathcal{O}_{\mathfrak{p}}^{\times}}$$
$$\chi_{\alpha_{\mathfrak{p}}}(u_{\mathfrak{p}}) = \alpha_{\mathfrak{p}}.$$

Now we are ready to prove the main result of this section.

Theorem 4.3. *For every element $\boldsymbol{v}$ of the tangent space of $\mathcal{U}$ at $\boldsymbol{k}$ we have*

$$\partial_{\boldsymbol{v}}(\chi_{\alpha_{\mathfrak{p}}}) \in \mathcal{L}_{\mathfrak{p}}(\pi)^{\epsilon}.$$

Proof. By Proposition 4.2, it is enough to prove that red_{χ} is surjective.

By Theorem 2.11, with the same arguments as in the proof of Proposition 2.13, the map induced in cohomology by (2.4)

$$\mathrm{H}^q_{\mathcal{O}(\mathcal{U}),\mathrm{cont}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\chi_{\boldsymbol{k}_{\mathfrak{p}}}), V^{\vee}_{\boldsymbol{k}^{\mathfrak{p}}})_{\mathfrak{m}_{\pi}^{S}} \longrightarrow \mathrm{H}^q(X_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, \mathcal{D}^n_{\mathcal{U},\boldsymbol{k}^{\mathfrak{p}}})_{(\mathfrak{m}_{\pi}^{S},U_p-1)}$$

is an isomorphism. Moreover, by Theorem 4.1, the reduction mod $\mathfrak{m}_{\boldsymbol{k}}$ map induces a surjective map

$$\mathrm{H}^q(X_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, \mathcal{D}^n_{\mathcal{U},\boldsymbol{k}^{\mathfrak{p}}})^{\epsilon}_{(\mathfrak{m}_{\pi}^{S},U_{\mathfrak{p}}-1)} \longrightarrow \mathrm{H}^q(X_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^{\vee})^{\epsilon}_{(\mathfrak{m}_{\pi}^{S},U_{\mathfrak{p}}-1)}.$$

On the other hand, one has the isomorphisms

$$\begin{aligned}\mathrm{H}^q(X^{\mathfrak{p}}_{K^{\mathfrak{p}}\times I_{\mathfrak{p}}}, V_{\boldsymbol{k}}(\Omega)^{\vee})^{\epsilon}_{(\mathfrak{m}_{\pi}^{S},U_{\mathfrak{p}}-1)} &\cong \mathrm{H}^q(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, i_B(\mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}}\\ &\cong \mathrm{H}^q_{\Omega,\mathrm{cont}}(X^{\mathfrak{p}}_{K^{\mathfrak{p}}}, \mathbb{I}_B(\mathbb{1}_{\boldsymbol{k}_{\mathfrak{p}}}), V_{\boldsymbol{k}^{\mathfrak{p}}}(\Omega)^{\vee})^{\epsilon}_{\mathfrak{m}_{\pi}^{S}}\end{aligned}$$

where the first isomorphism follows by the arguments in the proof of Proposition 2.13 and the second one from the non-criticality of π_{Ω}.

Recollecting all the maps, one gets the claim. □

4.4. **Relation with Galois representations.** Let $\rho = \rho_{\pi} \colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\Omega)$ be the 2-dimensional Galois representation attached to π and let $\rho_{\mathfrak{p}}$ be its restriction to a decomposition group $\mathrm{Gal}(\overline{F_{\mathfrak{p}}}/F_{\mathfrak{p}})$ at $\mathfrak{p}$. As local-global compatibility is known in this case by Saito (cf. [31]), the representation $\rho_{\mathfrak{p}}$ is semistable, non-crystalline, i.e.:

$$\mathcal{D}_{\mathrm{st}}(\rho_{\mathfrak{p}}) = (\rho_{\mathfrak{p}} \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{\mathrm{Gal}(\overline{F_{\mathfrak{p}}}/F_{\mathfrak{p}})}$$

is a free $\Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p},0}$-module (where $F_{\mathfrak{p},0}$ denotes the maximal unramified subfield of $F_{\mathfrak{p}}$) and the nilpotent linear map $N_{\mathfrak{p}}$ inherited from the corresponding map on Fontaine's semistable period ring B_{st} is non-zero. Moreover, the kernel of $N_{\mathfrak{p}}$ is a free $\Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p},0}$-module of rank one. Let e_0 be a generator of $\ker(N_{\mathfrak{p}})$ and define $e_1 = N_{\mathfrak{p}}(e_0)$. Furthermore, it is known that the zeroth step of the deRham filtration

$$\mathrm{Fil}^0(\mathcal{D}_{\mathrm{st}}(\rho_{\mathfrak{p}})) \subseteq \mathcal{D}_{\mathrm{st}}(\rho_{\mathfrak{p}}) \otimes_{F_{\mathfrak{p},0}} F_{\mathfrak{p}}$$

is a free $\Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p}}$-module of rank one. In particular there exist $a_0^{\rho_{\mathfrak{p}}}, a_1^{\rho_{\mathfrak{p}}} \in \Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p}}$ such that

$$\mathrm{Fil}^0(\mathcal{D}_{\mathrm{st}}(\rho_{\mathfrak{p}})) = a_0^{\rho_{\mathfrak{p}}} \cdot e_o + a_1^{\rho_{\mathfrak{p}}} \cdot e_1.$$

Definition 4.4. We call the local Galois representation $\rho_{\mathfrak{p}}$ non-critical if $a_0^{\rho_{\mathfrak{p}}} \in (\Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p}})^{\times}$.

It is expected that every $\rho_{\mathfrak{p}}$ coming from a Hilbert modular form as above is non-critical. If $F_{\mathfrak{p}} = \mathbb{Q}_p$ or $k_{\sigma} = 0$ for all $\sigma \in \Sigma_p$, the fact that $\mathcal{D}_{\mathrm{st}}(\rho_{\mathfrak{p}})$ is weakly admissible implies that $\rho_{\mathfrak{p}}$ is non-critical. It seems to the authors of this article that in both, [3], Section 5.2, and [12], Section 3.2, non-criticality of the Galois representation is assumed implicitly. Note that the main theorem of [38] states that a Galois representation as above is non-critical if $\mathfrak{p}$ is the only prime of F above p. But similar as on page 653 of [12] an Amice-Vélu and Vishik-type argument is used in the crucial Proposition 7.3 of *loc.cit.* and it is not clear to the authors of this article, if such an argument is applicable here (see Remark 2.6 above).

To any tuple

$$a = (a_\sigma) \in \Omega \otimes_{\mathbb{Q}_p} F_{\mathfrak{p}} \cong \prod_{\sigma \in \Sigma_{\mathfrak{p}}} \Omega$$

we attach a codimension one subspace

$$\mathcal{L}^a \subseteq \mathrm{Hom}(F_{\mathfrak{p}}^\times, \Omega)$$

that does not contain the subspace of smooth homomorphisms as follows: define

$$\log_\sigma = \log_{\mathfrak{p}} \circ \sigma \colon F_{\mathfrak{p}}^\times \to \Omega,$$

where $\log_{\mathfrak{p}}$ is the usual branch of the p-adic logarithm fulfilling $\log_p(p) = 0$ and put

$$\mathcal{L}^a = \langle \mathrm{ord}_{\mathfrak{p}} - a_\sigma \log_\sigma \mid \sigma \in \Sigma_{\mathfrak{p}} \rangle,$$

where $\mathrm{ord}_{\mathfrak{p}}$ denotes the normalized p-adic valuation of $F_{\mathfrak{p}}^\times$.

Definition 4.5. Suppose $\rho_{\mathfrak{p}}$ is non-critical. The Fontaine–Mazur $\mathcal{L}$-invariant of $\rho_{\mathfrak{p}}$ is the codimension one subspace

$$\mathcal{L}^{FM}(\rho_{\mathfrak{p}}) = \mathcal{L}^{a_1^{\rho_{\mathfrak{p}}}/a_0^{\rho_{\mathfrak{p}}}} \subseteq \mathrm{Hom}(F_{\mathfrak{p}}^\times, \Omega).$$

Theorem 4.6. *Suppose that π is non-critical at $\mathfrak{p}$ and that $\rho_{\mathfrak{p}}$ is non-critical. Then the equality*

$$\mathcal{L}_{\mathfrak{p}}(\pi)^\epsilon = \mathcal{L}(\rho_{\mathfrak{p}})^{FM}$$

holds for every sign character ϵ. In particular, the automorphic $\mathcal{L}$-invariant $\mathcal{L}_{\mathfrak{p}}(\pi)^\epsilon$ does not depend on the sign character ϵ.

Proof. This follows directly by comparing Theorem 4.3 with the corresponding formula on the Galois side (cf. [39], Theorem 1.1) for the family of Galois representation attached to the family passing through π. See [23], Theorem 4.1 for more details in case $k_\sigma = 0$ for all $\sigma \in \Sigma_{\mathfrak{p}}$. □

In [15] respectively [14] Ding proves that in case D is split at exactly one Archimedean place the Fontaine–Mazur $\mathcal{L}$-invariant can be detected by completed cohomology of the associated Shimura curve. Thus, by the theorem above the automorphic $\mathcal{L}$-invariant can also be detected by completed cohomology in that case. For the modular curve Breuil gives a direct proof of this consequence in [9]. It would be worthwhile to explore whether Breuil's proof extends to our more general setup.

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