EXTENSIONS OF THE STEINBERG REPRESENTATION

Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. We fix an embedding $F \hookrightarrow \mathbb{C}_p$. In Section 3.7 of [Spi14] Spieß constructs extensions of the Steinberg representation associated to characters of the multiplicative group $F^\times$. Such extensions were already constructed by Breuil in [Bre04] in case the character under consideration is a branch of the p-adic logarithm. After introducing a slightly improved version of Spieß' construction we will relate it to an extension class coming from the p-adic upper half plane. This in turn allows us to recast the uniformization of Jacobians of certain Mumford curves purely in representation theoretic terms.

Let A be a prodiscrete group. We define the A-valued (continuous) Steinberg representation of $\mathrm{PGL}_2(F)$ via

$$\mathrm{St}(A) = C(\mathbb{P}^1(F), A)/A.$$

We write $C_\diamond(F, A)$ for the space of continuous A-valued functions on F which vanish at infinity. The map

$$\delta\colon C_\diamond(F,A) \longrightarrow \mathrm{St}(A), \quad \delta(f)(P) = \begin{cases} f(x) & \text{if } P = [x:1], \\ 0 & \text{else,} \end{cases}$$

is an $F^\times$-equivariant isomorphism. Here $F^\times$ acts on $C_\diamond(F, A)$ via the left regular representation and on $\mathrm{St}(A)$ via the embedding

$$F^\times \longrightarrow \mathrm{GL}_2(F), \quad x \longmapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

Given a continuous group homomorphism $l\colon F^\times \to A$ we define $\widetilde{\mathcal{E}}(l)$ as the set of pairs $(\Phi, r) \in C(\mathrm{GL}_2(F), A) \times \mathbb{Z}$ with

$$\Phi\left(g \cdot \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix}\right) = \Phi(g) + r \cdot l(t_1)$$

for all $t_1, t_2 \in F^\times$, $u \in F$ and $g \in \mathrm{GL}_2(F)$. The group $\mathrm{GL}_2(F)$ acts on $\widetilde{\mathcal{E}}(l)$ via $g.(\Phi(\cdot), r) = (\Phi(g^{-1}\cdot), r)$. The subspace $\widetilde{\mathcal{E}}(l)_0$ of tuples of the form $(\Phi, 0)$ with constant Φ is $\mathrm{GL}_2(F)$-invariant. We get an induced $\mathrm{PGL}_2(F)$-action on the quotient $\mathcal{E}(l) = \widetilde{\mathcal{E}}(l)/\widetilde{\mathcal{E}}(l)_0$.

Lemma 1. *(a) Let $\pi\colon \mathrm{GL}_2(F) \to \mathbb{P}^1(F)$, $g \mapsto g.\infty$ be the canonical projection. The following sequence of $\mathbb{Z}[\mathrm{PGL}_2(F)]$-modules is exact:*

$$0 \longrightarrow \mathrm{St}_{\mathfrak{p}}(A) \xrightarrow{(\pi^*,0)} \mathcal{E}(l) \xrightarrow{(0,\mathrm{id}_{\mathbb{Z}})} \mathbb{Z} \longrightarrow 0$$

We write b_l for the associated cohomology class in $\mathrm{H}^1(\mathrm{PGL}_2(F), \mathrm{St}(A))$.

(b) Let $\delta^*\colon \mathrm{H}^1(\mathrm{PGL}_2(F), \mathrm{St}(A)) \to \mathrm{H}^1(F^\times, C_\diamond(F, A))$ be the homomorphism induced by the inverse of δ and let $c_l \in \mathrm{H}^1(F^\times, C_\diamond(F, A))$ be the class given by the cocycle

$$(z_l(a))(x) = l(a) \cdot \mathbb{1}_{a\mathcal{O}}(x) + l(x) \cdot (\mathbb{1}_{\mathcal{O}}(x) - \mathbb{1}_{a\mathcal{O}}(x)).$$

Then the equality $\delta^*(b_l) = c_l$ holds.

(c) The map

$$\mathcal{E}_A\colon \mathrm{Hom}_{\mathrm{cont}}(F^\times, A) \longrightarrow \mathrm{H}^1(\mathrm{PGL}_2(F), \mathrm{St}(A)), \quad l \longmapsto b_l$$

is a group homomorphism.

(d) Let $f\colon A \to A'$ be a continuous group homomorphism. The diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{cont}}(F^\times, A) \arrow[r, "\mathcal{E}_A"] \arrow[d, "f_*"'] & \mathrm{H}^1(\mathrm{PGL}_2(F), \mathrm{St}(A)) \arrow[d, "f_*"] \\
\mathrm{Hom}_{\mathrm{cont}}(F^\times, A') \arrow[r, "\mathcal{E}_{A'}"] & \mathrm{H}^1(\mathrm{PGL}_2(F), \mathrm{St}(A'))
\end{tikzcd}$$

is commutative.

Proof. (a): The only non-trivial step is to show that the map $\mathcal{E}(l) \xrightarrow{(0,\mathrm{id}_R)} R$ is surjective. A direct calculation shows that $(\Phi_0, 1)$ with

$$\Phi_0\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} l(a) & \text{if } \mathrm{ord}(a) < \mathrm{ord}(c), \\ l(c) & \text{if } \mathrm{ord}(a) \geq \mathrm{ord}(c), \end{cases}$$

defines an element of $\widetilde{\mathcal{E}}(l)$.

(b): A cocycle representing $\delta^*(b_l)$ is given by

$$\begin{aligned}(z_l'(a))(x) = &\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Phi_0\right)\left(\begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix}\right) - \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Phi_0\right)\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &-\Phi_0\left(\begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix}\right) + \Phi_0\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right).\end{aligned}$$

Plugging in the definition of Φ_0 one easily sees that $z_l = z_l'$.

(c): Let $l, l' \colon F^\times \to A$ be group homomorphisms. The Baer sum of the two extensions $\mathcal{E}(l)$ and $\mathcal{E}(l')$ is given as follows: we take the space of triples $(\overline{\Phi_1}, \overline{\Phi_2}, y)$ where $(\overline{\Phi_1}, y)$ (resp. $(\overline{\Phi_2}, y)$) is an element of $\mathcal{E}(l)$ (resp. $\mathcal{E}(l')$) modulo triples of the form $(\Phi, -\Phi, 0)$ with $\Phi \in \mathrm{St}_{\mathfrak{p}}(A)$. Sending a triple $(\overline{\Phi_1}, \overline{\Phi_2}, y)$ to the tuple $(\overline{\Phi_1 + \Phi_2}, y)$ defines a map from the Baer sum to $\mathcal{E}(l + l')$ and thus, they define the same extension class.

(d): Let $f_*(\mathcal{E}(l))$ be the pushout of the following diagram:

$$\begin{tikzcd}
\mathrm{St}_{\mathfrak{p}}(A) \arrow[r, "(\pi^*, 0)"] \arrow[d, "f_*"'] & \mathcal{E}(l) \\
\mathrm{St}_{\mathfrak{p}}(A') &
\end{tikzcd}$$

The homomorphism

$$\widetilde{\mathcal{E}}(l) \longrightarrow \widetilde{\mathcal{E}}(f \circ l),\ (\Phi, y) \longmapsto (f \circ \Phi, y),$$

induces a map from $f_*(\mathcal{E}(l))$ to $\mathcal{E}(f \circ l)$. Hence, they yield isomorphic extensions. $\square$

Remark 2. *The first two claims are essentially Lemma 3.11 of [Spi14]. The advantage of our version is that we get rid of the factor 2 showing up in Spieß' work, i.e. the cohomology class constructed above is "one half" of the cohomology class constructed in loc.cit. More precisely, Spieß considers the space $\mathcal{E}'(l)$ of tuples $(\Phi', r) \in C(\mathrm{GL}_2(F), A) \times \mathbb{Z}$ with*

$$\Phi'\left(g \cdot \begin{pmatrix} t_1 & u \\ 0 & t_2 \end{pmatrix}\right) = \Phi'(g) + r \cdot l(t_1/t_2)$$

modulo tuples of the form $(\Phi', 0)$ with Φ' constant. The map sending (Φ', r) to $(\Phi' + l \circ \det, 2r)$ induces a $\mathrm{PGL}_2(F)$-equivariant isomorphism between $\mathcal{E}'(l)$ and the subspace of those tuples (Φ, r) in $\mathcal{E}(l)$ with r even.

For every totally disconnected compact space X there is an integration pairing

$$C(X,A)\otimes \operatorname{Dist}(X,\mathbb{Z})\longrightarrow A \tag{1}$$

constructed as follows: For any discrete group B the canonical map

$$C(X,\mathbb{Z})\otimes B \xrightarrow{\cong} C(X,B)$$

is an isomorphism and, thus, we have a canonical pairing

$$C(X,B)\otimes \operatorname{Dist}(X,\mathbb{Z})\longrightarrow B$$

that is functorial in B. Let us choose a basis of neighbourhoods $\{K_i\}$ of the identity of A consisting of open subgroups with $K_{i+1}\subseteq K_i$. Taking projective limits we get a pairing

$$\varprojlim_i C(X,A/K_i)\otimes \operatorname{Dist}(X,\mathbb{Z})\longrightarrow \varprojlim_i A/K_i = A.$$

The canonical map

$$C(X,A)\xrightarrow{\cong}\varprojlim_i C(X,A/K_i)$$

is an isomorphism and hence, we constructed the desired pairing. In particular, given a subgroup $\Gamma\subseteq \mathrm{PGL}_2(F)$ and a prodiscrete group A the integration pairing induces a cup product pairing

$$\mathrm{H}^i(\Gamma,\operatorname{St}(\mathbb{Z})^\vee)\otimes \mathrm{H}^j(\Gamma,\operatorname{St}(A))\xrightarrow{\cup}\mathrm{H}^{i+j}(\Gamma,A) \tag{2}$$

in cohomology.

Let $b_{\mathrm{univ}}\in \mathrm{H}^1(\mathrm{PGL}_2(F),\operatorname{St}(F^\times))$ be the cohomology class associated to the identity $F^\times\to F^\times$. We will relate it to a class obtained from the p-adic upper half plane. For a closed subfield $E\subset\mathbb{C}_p$ containing F we put $\mathcal{H}_F(E)=\mathbb{P}^1(E)-\mathbb{P}^1(F)$. The group $\mathrm{PGL}_2(F)$ acts on $\mathcal{H}_F(E)$ via Möbius transformations and thus by linear extension on the space $\operatorname{Div}(\mathcal{H}_F(E))$ of (naive) divisors on $\mathcal{H}_F(E)$, i.e. the space of formal $\mathbb{Z}$-linear combinations of points of $\mathcal{H}_F(E)$. The subspace $\operatorname{Div}^0(\mathcal{H}_F(E))$ of divisors of degree 0 is $\mathrm{PGL}_2(F)$-invariant. Sending a degree 0 divisor $D=\sum_{z\in\mathcal{H}_F(E)} n_z[z]$ to the map $x\mapsto\prod_{z\in\mathcal{H}_F(E)}(x-z)^{n_z}$, i.e. the unique (up to multiplication by a constant) rational function with divisor D, yields a $\mathrm{PGL}_2(F)$-equivariant homomorphism

$$\Psi\colon \operatorname{Div}^0(\mathcal{H}_F(E))\longrightarrow \operatorname{St}(E^\times).$$

Lemma 3. *Let $b_{\mathrm{geom}}\in \mathrm{H}^1(\mathrm{PGL}_2(F),\operatorname{Div}^0(\mathcal{H}_F(E)))$ be the class of the extension*

$$0\longrightarrow \operatorname{Div}^0(\mathcal{H}_F(E))\longrightarrow \operatorname{Div}(\mathcal{H}_F(E))\xrightarrow{\deg}\mathbb{Z}\longrightarrow 0. \tag{3}$$

The equality $\Psi_*(b_{\mathrm{geom}})=b_{\mathrm{univ}}$ holds in $\mathrm{H}^1(\mathrm{PGL}_2(F),\operatorname{St}(E^\times))$.

Proof. For $z\in\mathcal{H}_F(E)$ we define the function

$$F_z\colon \mathrm{GL}_2(F)\longrightarrow E^\times,\quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\longmapsto cz-a.$$

A short calculation shows that the homomorphism

$$\widetilde{\Psi}\colon \operatorname{Div}(\mathcal{H}_F(E))\longrightarrow \mathcal{E}(\iota),\quad \sum_z n_z[z]\longmapsto \Big(\prod_z (F_z)^{n_z},\sum_z n_z\Big)$$

is $\mathrm{PGL}_2(F)$-equivariant, where $\iota\colon F^\times\to E^\times$ denotes the embedding. One immediately checks that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Div}^0(\mathcal{H}_F(E)) \arrow[r] \arrow[d, "\Psi"] & \mathrm{Div}(\mathcal{H}_F(E)) \arrow[r] \arrow[d, "\widetilde{\Psi}"] & \mathbb{Z} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathrm{St}(E^\times) \arrow[r] & \mathcal{E}(\iota) \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

is commutative and therefore, the claim follows. □

A discrete subgroup $\Gamma \subset \mathrm{PGL}_2(F)$ is called a (p-adic) Schottky group if it is finitely generated and torsion-free. A Schottky group acts properly discontinuously on the Bruhat-Tits tree of $\mathrm{PGL}_2(F)$ and, therefore, it is a free group (see for example [Ser03], Chapter I.3.3) It follows that for a Schottky group Γ and any abelian group A the canonical map

$$\mathrm{H}^1(\Gamma,\mathbb{Z})\otimes A \longrightarrow \mathrm{H}^1(\Gamma, A) \tag{4}$$

is an isomorphism. We define the algebraic torus $\mathbb{G}_\Gamma = \mathrm{H}^1(\Gamma,\mathbb{Z})\otimes\mathbb{G}_m$. It follows from (4) that we have a canonical isomorphism $\mathbb{G}(E)\cong \mathrm{H}^1(\Gamma, E^\times)$ for all extensions E of F. The cup product pairing (2) induces a homomorphism

$$\int_{\mathrm{univ}}\colon\ \mathrm{H}^0(\Gamma,\mathrm{Hom}(\mathrm{St}(\mathbb{Z}),\mathbb{Z})) \xrightarrow{\cup b_{\mathrm{univ}}} \mathrm{H}^1(\Gamma,\mathbb{C}_p^\times) = \mathbb{G}_\Gamma(\mathbb{C}_p).$$

We define L_Γ to be its image.

Let X be a smooth proper curve over F with Jacobian Jac_X. We assume that X admits a uniformization by the p-adic upper half plane, i.e. there exists a Schottky group Γ_X and a $\mathrm{Gal}(\mathbb{C}_p/F)$-equivariant rigid analytic isomorphism

$$X(\mathbb{C}_p)\cong \Gamma_X\backslash\mathcal{H}_F(\mathbb{C}_p).$$

Proposition 4 (p-adic uniformization)**.** *There is a* $\mathrm{Gal}(\mathbb{C}_p/F)$*-equivariant rigid analytic isomorphism*

$$\mathrm{Jac}_X(\mathbb{C}_p)\cong \mathbb{G}_{\Gamma_X}(\mathbb{C}_p)/L_{\Gamma_X}.$$

Proof. Let

$$\delta\colon\ \mathrm{H}_1(\Gamma_X,\mathbb{Z}) \longrightarrow \mathrm{H}_0(\Gamma_X,\mathrm{Div}^0(\mathcal{H}_F(\mathbb{C}_p)))$$

be the boundary map coming from the short exact sequence (3) and let

$$⨍\colon\ \mathrm{H}_0(\Gamma_X,\mathrm{Div}^0(\mathcal{H}_F(\mathbb{C}_p))) \longrightarrow \mathbb{G}_{\Gamma_X}(\mathbb{C}_p)$$

be the multiplicative integral introduced in Section 2 of [Das05]. We define $\widetilde{L}_{\Gamma_X}$ to be the image of the composition of the above maps. By Dasgupta's variant of the Manin-Drinfeld Theorem there is a rigid analytic isomorphism

$$\mathrm{Jac}_X(\mathbb{C}_p)\cong \mathbb{G}_{\Gamma_X}(\mathbb{C}_p)/\widetilde{L}_{\Gamma_X}.$$

It follows from Lemma 3 that the diagram

$$\begin{tikzcd}
\mathrm{H}_1(\Gamma_X,\mathbb{Z}) \arrow[r, "\delta"] \arrow[d, "="] & \mathrm{H}_0(\Gamma_X,\mathrm{Div}^0(\mathcal{H}_F(\mathbb{C}_p))) \arrow[r, "⨍"] \arrow[d, "\Psi_*"] & \mathbb{G}_{\Gamma_X}(\mathbb{C}_p) \arrow[d, "="] \\
\mathrm{H}_1(\Gamma_X,\mathbb{Z}) \arrow[r, "\delta_{\mathrm{univ}}"] & \mathrm{H}_0(\Gamma_X,\mathrm{St}(\mathbb{C}_p^\times)) \arrow[r, "⨍"] & \mathbb{G}_{\Gamma_X}(\mathbb{C}_p)
\end{tikzcd}$$

is commutative, where δ_{univ} denotes the boundary map coming from the short exact sequence corresponding to b_{univ}.

Since Γ_X acts freely on the Bruhat-Tits tree $\mathcal{T}$ of $\mathrm{PGL}_2(F)$ it follows that $\mathrm{St}(\mathbb{Z}) = \mathrm{H}^1_c(\mathcal{T}, \mathbb{Z})$ is a dualizing module for Γ_X. In particular, $\mathrm{H}_1(\Gamma_X, \mathrm{St}(\mathbb{Z}))$ is a free $\mathbb{Z}$-module of rank one and taking cap product with a generator induces an isomorphism

$$\mathrm{D}\colon\ \mathrm{H}^0(\Gamma_X, \mathrm{Hom}(\mathrm{St}(\mathbb{Z}), \mathbb{Z})) \longrightarrow \mathrm{H}_1(\Gamma_X, \mathbb{Z}).$$

A tedious but formal computation in cohomology shows that

$$\int_{\mathrm{univ}} = \pm \times\!\!\!\!\!\!\int \circ\, \delta_{\mathrm{univ}} \circ \mathrm{D}.$$

□

References

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