Automorphic L-Invariants

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Structure of talk

1. P-adic periods of elliptic curves
2. Automorphic L-Invariants

Elliptic curves

k field, $char(k) \neq 2, 3$
E/k elliptic curve, $E = V(Y^2 - X^3 - aX - b)$ with $a, b \in k$ such that $4a^3 + 27b^2 \neq 0$

Examples: ($k = \mathbb{R}$)

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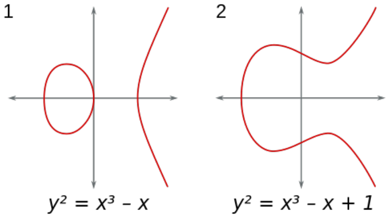

Group law

E/k is a projective algebraic group
$\rightsquigarrow E(k)$ is an (abelian) group

For $P, Q, R \in E(k)$:

$$P + Q + R = 0 \text{ iff } P, Q, R \text{ are colinear}$$

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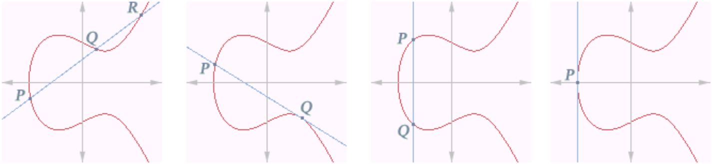

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Complex uniformization

$k = \mathbb{C}$

$\rightsquigarrow E(\mathbb{C})$ is a complex one-dimensional torus, i.e.:

$$E(\mathbb{C}) \cong \mathbb{C}/\Lambda, \ \Lambda \subseteq \mathbb{C} \text{ lattice}$$

Wlog $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, $\mathfrak{Im}(\tau) > 0$

$$\mathbb{C}/\mathbb{Z} \xrightarrow{\exp(2\pi i\ \cdot)} \mathbb{C}^*$$

yields

$$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}, \ 0 < |q| < 1$$

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What about $k = \mathbb{Q}_p$?

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$$\rightsquigarrow E \cong E_q,\ q \in \mathbb{Q}_p,\ 0 < |q|_p < 1$$

Modularity

$k = \mathbb{Q}$

(Wiles et al.) $\rightsquigarrow \exists\, f \in S_2(\Gamma_0(N))$ such that

$$L(E, s) = L(f, s)$$

f is a differential form on a Riemann surface
$\rightsquigarrow$ purely analytic object

Suppose E has split multiplicative reduction at prime p
(Tate) $\rightsquigarrow$ $q_E \in \mathbb{Q}_p^*$

Question: Can we read q_E off of f?

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L-Invariants

Answer:

$E \sim E'$ isogenous $\rightsquigarrow L(E,s) = L(E',s)$
but: $q_E^n = q_{E'}^m$ for $m, n \in \mathbb{Z}_{>0}$

$$\rightsquigarrow \mathcal{L}_p(E) = \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)} \text{ isogeny invariant}$$

Question: Can we read $\mathcal{L}_p(E)$ off of f?

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1. P-adic periods of elliptic curves

2. Automorphic L-Invariants

Cohomology of arithmetic groups

$E \longleftrightarrow f$ weight 2 modular form of level N

$$\Gamma_0(N) = \left\{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}$$

$\rightsquigarrow f$ (holomorphic) 1-form on $\Gamma_0(N) \backslash \mathbb{H}$

$$f \in H^1(\Gamma_0(N) \backslash \mathbb{H}, \mathbb{C}) \cong H^1(\Gamma_0(N), \mathbb{C})$$

Multiplicity one theorem:

$$\dim H^1(\Gamma_0(N), \mathbb{C})^+[f] = 1$$
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Cohomology of S-arithmetic groups

E split multiplicative reduction at *p*

$$\Gamma_0^p(N) = \left\{ \gamma \in \mathrm{GL}_2(\mathbb{Z}[1/p]) \mid \gamma \cong \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N,\ \det(\gamma) > 0 \right\}$$

$$\mathrm{St}_p(\mathbb{Q}_p) = \left\{ \phi \colon \mathbb{P}^1(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p \mid \phi \text{ locally constant} \right\} / \{\text{constant}\}$$

There is a natural isomorphism of 1-dim vector spaces:

$$\mathrm{H}^1(\Gamma_0^p(N), \mathrm{St}_p(\mathbb{Q}_p)^\vee)^+[f] \longrightarrow \mathrm{H}^1(\Gamma_0(N), \mathbb{Q}_p)^+[f]$$

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Cohomology of S-arithmetic groups

E split multiplicative reduction at p

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P-adic integration

Remember $\mathcal{L}_p(E) = \frac{\log_p(q_E)}{\mathrm{ord}_p(q_E)}$

$\rightsquigarrow$ quotient of two additive characters

Let $\lambda\colon \mathbb{Q}_p^* \longrightarrow \mathbb{Q}_p$ additive character

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$$c_{\log_p} = \qquad\qquad \cdot\, c_{\mathrm{ord}_p}$$

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Equality of L-Invariants

Theorem (Greenberg-Stevens 93, Darmon 01)

$$\mathcal{L}_p^{\mathrm{aut}}(E) = \mathcal{L}_p(E)$$

- Darmon: $\mathcal{L}_p^{\mathrm{aut}}(E) =$ derivative of p-adic L-function
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Question: Is there an easier proof?

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Theorem (G 17)

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