The Lifted Root Number Conjecture
for small sets of places
and an application to CM-extensions

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   - Galois module structure

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The Riemann zeta function

The Riemann zeta function

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \quad \Re(s) > 1
\]

has an analytical continuation to \( \mathbb{C} \setminus \{1\} \) and a simple pole at \( s = 1 \) with residue 1.

The Riemann zeta function is attached to the number field \( \mathbb{Q} \) of rational numbers.
The Dedekind zeta function I

- Let $L$ be any number field
- $\mathcal{o}_L = \text{ring of integers of } L$
Let $L$ be any number field

- $\mathfrak{o}_L = \text{ring of integers of } L$

The Dedekind zeta function

\[ \zeta_L(s) = \prod_{\mathfrak{p} \in \mathfrak{o}_L \text{ prime}} \frac{1}{1 - N(\mathfrak{p})^{-s}}, \quad \Re(s) > 1 \]

is attached to $L$, where $N(\mathfrak{p}) = \lvert \mathfrak{o}_L/\mathfrak{p} \rvert$. 
The Dedekind zeta function II

- $\zeta_L(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$ and a simple pole at $s = 1$ with residue

$$2^{r_1}(2\pi)^{r_2} h_L R_L \over \omega_L \sqrt{|d_L|}.$$

- Here, $r_1$ resp. $r_2$ denotes the number of real resp. pairs of complex conjugate embeddings of $L$, $\omega_L$ is the number of roots of unity in $L$, $d_L$ the discriminant, $R_L$ the regulator and $h_L$ the class number of $L$. 
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The structure of units

- Aim: Describe the structure of the units $\mathfrak{o}_L^\times$.

**Theorem (Dirichlet's unit theorem)**

Write $\mu_L$ for the roots of unity of $L$. Then we have an isomorphism of $\mathbb{Z}$-modules

$$\mathfrak{o}_L^\times \cong \mu_L \times \mathbb{Z}^{r_1+r_2-1}.$$
The structure of units

- **Aim**: Describe the structure of the units $\mathfrak{O}_L^\times$.

**Theorem (Dirichlet’s unit theorem)**

Write $\mu_L$ for the roots of unity of $L$. Then we have an isomorphism of $\mathbb{Z}$-modules

$$\mathfrak{O}_L^\times \simeq \mu_L \times \mathbb{Z}^{r_1+r_2-1}.$$ 

- Whenever $L/K$ is a Galois extension of number fields with Galois group $G$, each $\sigma \in G$ maps units to units; hence $G$ acts on $\mathfrak{O}_L^\times$.
- **Higher aim**: Describe $\mathfrak{O}_L^\times$ as a $\mathbb{Z}G$-module!
- Many further objects attached to $L$ are equipped with a natural action of $G$. 
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**S-units**

- $S_\infty = \text{set of all infinite primes, i.e. of all real embeddings and all pairs of complex conjugate embeddings of } L$
- $S = S_\infty \cup \text{finite } G\text{-invariant set of prime ideals of } \mathfrak{o}_L$
- $S$ is called a finite set of places of $L$. 

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**Motivation**

- The Conjecture
- CM-extensions

**The arithmetic side**

- Proven cases
- Small sets of places

**The analytic side**

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LRCN
$S$-units

- $S_\infty = \text{set of all infinite primes}$, i.e. of all real embeddings and all pairs of complex conjugate embeddings of $L$
- $S = S_\infty \cup \text{finite } G\text{-invariant set of prime ideals of } \mathfrak{o}_L$
- $S$ is called a finite set of places of $L$.
- We want to admit denominators and define

$$\mathfrak{o}_S = \left\{ \frac{a}{b} \middle| a, b \in \mathfrak{o}_L, \mathfrak{p} \nmid (b) \forall \mathfrak{p} \not\in S \right\}.$$

- The $S$-units $E_S$ are the invertible elements of $\mathfrak{o}_S$: $E_S := \mathfrak{o}_S^\times$. 
The Tate-sequence

If $S$ is “sufficiently” large, there exists an exact sequence ("Tate-sequence")

$$E_S \rightarrow A \rightarrow B \rightarrow \Delta S$$

with a uniquely determined extension class in $\text{Ext}^2_G(\Delta S, E_S)$;
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- $\Delta S$ is the kernel of the augmentation-map

$$\mathbb{Z}S \rightarrow \mathbb{Z}, \quad \sum \chi_p \mathfrak{p} \mapsto \sum \chi_p.$$

- $\Delta S$ has easy $G$-module structure.
The Tate-sequence

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\[
\mathbb{Z}S \rightarrow \mathbb{Z}, \quad \sum \mathfrak{p} \mathfrak{P} \mapsto \sum \mathfrak{p} \mathfrak{P}.
\]

- \( \Delta S \) has easy \( G \)-module structure.
- \( B \) is \( \mathbb{Z}G \)-projective and can be chosen \( \mathbb{Z}G \)-free.
- \( A \) is cohomologically trivial, i.e. there is a short exact sequence

\[
P_1 \hookrightarrow P_0 \twoheadrightarrow A
\]

with \( \mathbb{Z}G \)-projective \( P_0 \) and \( P_1 \).
The arithmetic object

- There exist equivariant embeddings \( \phi: \Delta S \hookrightarrow E_S \) with finite cokernel.
- One can transpose \( \phi \) to an embedding \( \tilde{\phi}: B \hookrightarrow A \) with finite cokernel; this cokernel is cohomologically trivial.
The arithmetic object

- There exist equivariant embeddings \( \phi : \Delta S \hookrightarrow E_S \) with finite cokernel.
- One can transpose \( \phi \) to an embedding \( \tilde{\phi} : B \hookrightarrow A \) with finite cokernel; this cokernel is cohomologically trivial.
- One defines (K.W. Gruenberg, J. Ritter, A. Weiss 1999)

  \[
  \Omega_{\phi} := (\text{cok } \tilde{\phi}) - \text{ correction term } \in K_0 T(\mathbb{Z}G).
  \]

- \( K_0 T(\mathbb{Z}G) \) is the free abelian group generated by isomorphism classes of finite cohomologically trivial \( \mathbb{Z}G \)-modules modulo short exact sequences.
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   - Related conjectures
Let $L \subset F$ be a number field such that $F/\mathbb{Q}$ is a Galois extension with Galois group $\Gamma$ and “sufficiently” large. Let $\chi$ be a character of $G$. 
Let $L \subset F$ be a number field such that $F/\mathbb{Q}$ is a Galois extension with Galois group $\Gamma$ and “sufficiently” large. Let $\chi$ be a character of $G$.

- $V_\chi = \text{the } FG\text{-module attached to } \chi$.
- $\check{V}_\chi = \text{Hom}_F(V_\chi, F) = \text{contragredient of } V_\chi \text{ with character } \check{\chi}$.
- $R(G) = \text{free abelian group generated by the irreducible characters of } G$. 
The map

\[ R\phi : R(G) \rightarrow \mathbb{C}^\times \]

\[ \chi \mapsto \det(\lambda_S \circ \phi|\text{Hom}_G(\tilde{\mathcal{V}}_\chi, \mathbb{C} \otimes \Delta S)) \]

is called the **Stark-Tate regulator**, where

\[ \lambda_S : E_S \rightarrow \mathbb{R} \otimes \Delta S \]

\[ \varepsilon \mapsto - \sum_{\mathfrak{p} \in S} \log |\varepsilon|_{\mathfrak{p}} \]

is the **Dirichlet map**.
For a prime ideal $\mathfrak{P}$ of $L$ let $\phi_{\mathfrak{P}} \in G$ be the Frobenius automorphism and $I_{\mathfrak{P}} \leq G$ the inertia subgroup at $\mathfrak{P}$. The series

$$L_S(L/K, \chi, s) = \prod_{p \notin S(K)} \det(1 - \phi_{\mathfrak{P}} N(p)^{-s} | V_{\chi}^{I_{\mathfrak{P}}})^{-1},$$

converges for $s \in \mathbb{C}, \Re(s) > 1$. 
For a prime ideal $\mathfrak{P}$ of $L$ let $\phi_{\mathfrak{P}} \in G$ be the Frobenius automorphism and $I_{\mathfrak{P}} \leq G$ the inertia subgroup at $\mathfrak{P}$. The series

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converges for $s \in \mathbb{C}, \Re(s) > 1$.

- $L_S(L/K, \chi, s)$ has a meromorphic continuation to $\mathbb{C}$.
- We denote the leading coefficient of the Taylor expansion at $s = 0$ by $c_S(\chi)$.
**L-functions**

For a prime ideal \( \mathfrak{p} \) of \( L \) let \( \phi_{\mathfrak{p}} \in G \) be the Frobenius automorphism and \( I_{\mathfrak{p}} \leq G \) the inertia subgroup at \( \mathfrak{p} \). The series

\[
L_S(L/K, \chi, s) = \prod_{p \not\in S(K)} \det(1 - \phi_{\mathfrak{p}} N(p)^{-s} | V_{\chi \mathfrak{p}} )^{-1},
\]

converges for \( s \in \mathbb{C}, \Re(s) > 1 \).

- \( L_S(L/K, \chi, s) \) has a meromorphic continuation to \( \mathbb{C} \).
- We denote the **leading coefficient** of the Taylor expansion at \( s = 0 \) by \( c_S(\chi) \).

**Example**

For \( L = K, S = S_\infty \) and \( \chi = 1 \) we have \( L_{S_\infty}(K/K, 1, s) = \zeta_K(s) \).

Especially for \( K = \mathbb{Q} \), we have

\[
L_{S_\infty}(\mathbb{Q}/\mathbb{Q}, 1, s) = \zeta(s), \quad c_{S_\infty}(1) = \zeta(0) = -\frac{1}{2}.
\]
Conjecture (Stark)

The map

\[ R(G) \rightarrow \mathbb{C}^\times \]
\[ \chi \mapsto \frac{R_\phi(\bar{\chi})}{c_S(\bar{\chi})} W(\bar{\chi}) \]

lies in \( \text{Hom}_\Gamma(R(G), F^\times) \). Here, \( W(\bar{\chi}) = \pm 1 \) is defined via Artin root numbers.
**Stark’s Conjecture and the LRNC**

**Conjecture (Stark)**

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If Stark’s Conjecture holds, the above homomorphism defines an element \( \Theta_{\phi} \in K_0 T(\mathbb{Z} G) \).
Stark’s Conjecture and the LRNC

Conjecture (Stark)

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\begin{align*}
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If Stark’s Conjecture holds, the above homomorphism defines an element \( \Theta_{\phi} \in K_0 T(\mathbb{Z} G) \).

Conjecture (LRNC, K.W. Gruenberg, J. Ritter, A. Weiss 1999)

\( \Theta_{\phi} = \Omega_{\phi} \).
Choose a maximal order $\mathcal{M}$: $\mathbb{Z}G \subset \mathcal{M} \subset \mathbb{Q}G$. There is a natural map

$$\delta_{\mathcal{M}} : K_0 T(\mathbb{Z}G) \to K_0 T(\mathcal{M})$$

induced by $\otimes_{\mathbb{Z}G} \mathcal{M}$. 

Conjecture (Strong Stark)
Assume that Stark’s Conjecture holds. Then

$$\delta_{\mathcal{M}}(\Theta \varphi) = \delta_{\mathcal{M}}(\Omega \varphi)$$

This conjecture is independent of the choice of $\mathcal{M}$. 

Andreas Nickel
LRNC
Strong Stark Conjecture

Choose a maximal order $\mathcal{M}$: $\mathbb{Z}G \subset \mathcal{M} \subset \mathbb{Q}G$. There is a natural map

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**Conjecture (Strong Stark)**

*Assume that Stark’s Conjecture holds. Then*

$$\delta_\mathcal{M}(\Theta_\phi) = \delta_\mathcal{M}(\Omega_\phi).$$
Choose a maximal order $\mathcal{M}$: $\mathbb{Z} G \subset \mathcal{M} \subset \mathbb{Q} G$. There is a natural map

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*LRNC*
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Proven cases

- $K = \mathbb{Q}$ and $L/\mathbb{Q}$ abelian (D. Burns, C. Greither 2003; for $L$ totally real also by J. Ritter, A. Weiss 2002/03)
Proven cases

- $K = \mathbb{Q}$ and $L/\mathbb{Q}$ abelian (D. Burns, C. Greither 2003; for $L$ totally real also by J. Ritter, A. Weiss 2002/03)

- $L/K$ with $E \subseteq K \subseteq L$ such that $E = \mathbb{Q}(\sqrt{d})$, where $d < 0$, $h_E = 1$ and $L/E$ is abelian, and such that $[L : K]$ is odd and divisible only by primes which split completely in $E/\mathbb{Q}$. (W. Bley 2006)
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A Tate sequence for small sets of places

- Actually, we are interested in the units: $\mathfrak{o}_L^\times = E_{S_\infty}$.
- Let the **only** constraint on $S$ be: $S_\infty \subset S$. Then there is a Tate sequence (J. Ritter, A. Weiss 1996)

$$E_S \rightarrowtail A \twoheadrightarrow B \twoheadrightarrow \nabla.$$
A Tate sequence for small sets of places

- Actually, we are interested in the units: $\mathfrak{o}_L^\times = E_S\infty$.
- Let the only constraint on $S$ be: $S\infty \subset S$. Then there is a Tate sequence (J. Ritter, A. Weiss 1996)
  $$E_S \hookrightarrow A \rightarrow B \rightarrow \nabla.$$  

- The torsion submodule of $\nabla$ is the $S$ class group $\text{cl}_S(L)$:
  $$\text{cl}_S(L) \hookrightarrow \nabla \rightarrow \overline{\nabla}$$
  where $\overline{\nabla}$ is $\mathbb{Z}$-free.
A Tate sequence for small sets of places

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- The **torsion submodule** of $\nabla$ is the $S$ class group $\text{cl}_S(L)$:

$$\text{cl}_S(L) \hookrightarrow \nabla \rightarrow \overline{\nabla}$$

where $\overline{\nabla}$ is $\mathbb{Z}$-free.
- $\overline{\nabla}$ is the kernel in an exact sequence

$$\overline{\nabla} \hookrightarrow \mathbb{Z}S \bigoplus \bigoplus_{\mathfrak{P} \in S^*_\text{ram} \setminus (S \cap S^*_\text{ram})} \text{ind} \frac{G_{\mathfrak{P}}}{G_{\mathfrak{P}}} W^0_{\mathfrak{P}} \rightarrow \mathbb{Z}$$

with explicit $\mathbb{Z}G_{\mathfrak{P}}$-modules $W^0_{\mathfrak{P}}$. 
Main problem: In general there are no embeddings \( \phi : \nabla \rightarrow E_S \).
The conjecture for small sets of places I

- Main problem: In general there are no embeddings $\phi : \nabla \hookrightarrow E_S$.
- Solution: There exist $\mathbb{Q}G$-isomorphisms

$$\phi : \mathbb{Q} \otimes \nabla \xrightarrow{\sim} \mathbb{Q} \otimes (E_S \oplus C)$$

with a $\mathbb{Z}G$-free module $C$ of suitable rank.
The conjecture for small sets of places I

- **Main problem:** In general there are no embeddings \( \phi : \nabla \hookrightarrow E_S \).

- **Solution:** There exist \( \mathbb{Q}G \)-isomorphisms

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\phi : \mathbb{Q} \otimes \nabla \xrightarrow{\sim} \mathbb{Q} \otimes (E_S \oplus C)
\]

with a \( \mathbb{Z}G \)-free module \( C \) of suitable rank.

- **From this one can construct** \( \mathbb{Q}G \)-isomorphisms

\[
\tilde{\phi} : \mathbb{Q} \otimes B \xrightarrow{\sim} \mathbb{Q} \otimes (A \oplus C).
\]

- **Define**

\[
\Omega_\phi := (B, \tilde{\phi}, A \oplus C) - \text{correction term} \in K_0(\mathbb{Z}G, \mathbb{Q}) \simeq K_0 T(\mathbb{Z}G).
\]
The conjecture for small sets of places II

- Determine how $\Omega_\phi$ varies if $S$ is enlarged by (orbits of) prime ideals.
- Most interesting (and difficult to handle) are the primes which ramify in $L/K$.
- Unramified primes behave as before.
Determine how $\Omega_\phi$ varies if $S$ is enlarged by (orbits of) prime ideals.

Most interesting (and difficult to handle) are the primes which ramify in $L/K$.

Unramified primes behave as before.

This leads to the definition of a modified Stark-Tate regulator $R^{\text{mod}}_\phi$. 
Conjecture (LRNC for small $S$)

The element $\Omega_\phi \in K_0(\mathbb{Z}G, \mathbb{Q})$ is represented by the homomorphism

$$R(G) \rightarrow \mathbb{C}^\times, \quad \chi \mapsto \frac{R_\phi^{\text{mod}}(\check{\chi})}{c_{S \cup S_{\text{ram}}}(\check{\chi})} W(\check{\chi}).$$
The conjecture for small sets of places II

Conjecture (LRNC for small $S$)

The element $\Omega_\phi \in K_0(\mathbb{Z}G, \mathbb{Q})$ is represented by the homomorphism

$$R(G) \rightarrow \mathbb{C}^\times, \chi \mapsto \frac{R^{\mod}(\tilde{\chi})}{c_{S \cup S_{\text{ram}}}(\tilde{\chi})} W(\tilde{\chi}).$$

- The LRNC for small sets of places is equivalent to the LRNC for large sets of places.
- The LRNC naturally decomposes in local conjectures at each prime $p$ by means of the isomorphism

$$K_0(\mathbb{Z}G, \mathbb{Q}) \simeq \bigoplus_p K_0(\mathbb{Z}_p G, \mathbb{Q}_p)$$
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   • Related conjectures
Let $L/K$ be a Galois **CM-extension** with Galois group $G$, i.e.

- $L$ is totally complex
- $K$ is totally real
- **Complex conjugation** defines an automorphism $j \in G$ on $L$, central in $G$:
  
  $$jg = gj \quad \forall g \in G$$
CM-extensions

Let \( L/K \) be a Galois CM-extension with Galois group \( G \), i.e.

- \( L \) is totally complex
- \( K \) is totally real
- **Complex conjugation** defines an automorphism \( j \in G \) on \( L \), central in \( G \):

\[
jg = gj \quad \forall g \in G
\]

**Example**

The extensions \( \mathbb{Q}(\zeta_n)/\mathbb{Q} \) are abelian CM-extensions, where \( \zeta_n \) denotes a \( n \)-th root of unity. Each abelian extension of \( \mathbb{Q} \) lies in such an extension.
For a $G$-Modul $M$ set

$$M^\pm = \{ m \in M | jm = \pm m \}.$$ 

If $M$ is a $\mathbb{Z}_p G$-module and $p \neq 2$, there is a natural decomposition

$$M = M^+ \oplus M^-.$$
Plus and minus parts

- For a $G$-Modul $M$ set
  \[
  M^\pm = \{ m \in M \mid jm = \pm m \}.
  \]

- If $M$ is a $\mathbb{Z}_pG$-module and $p \neq 2$, there is a natural decomposition
  \[
  M = M^+ \oplus M^-.
  \]

- Accordingly, the LRNC decomposes into a **plus** and a **minus** part.
For a $G$-Modul $M$ set

$$M^{\pm} = \{ m \in M | jm = \pm m \}.$$ 

If $M$ is a $\mathbb{Z}_p G$-module and $p \neq 2$, there is a natural decomposition

$$M = M^+ \oplus M^-.$$ 

Accordingly, the LRNC decomposes into a plus and a minus part.
Stark’s conjecture is known to be true on minus parts.
Let $T$ be a finite set of prime ideals of $L$. Then we denote by

$$\text{cl}_L^T := \frac{\{\text{fractional ideals of } L, \text{ coprime to all } \mathfrak{p} \in T\}}{\{(a) | a \in L, \ a \equiv 1 \mod \mathfrak{p} \ \forall \mathfrak{p} \in T\}}$$

the ray class group of $L$ to the ray $\mathcal{M}_T := \prod_{\mathfrak{p} \in T} \mathfrak{p}$.
Ray class groups II

**Theorem**

Let $L/K$ be a Galois CM-extension with Galois group $G$ and $p \neq 2$ a prime such that $L/K$ is at most tamely ramified above $p$. Then there exist finite sets $T$ of primes of $L$ such that $(c_L^T)^- \otimes \mathbb{Z}_p$ is cohomologically trivial.
Theorem

Let $L/K$ be a Galois CM-extension with Galois group $G$ and $p \neq 2$ a prime such that $L/K$ is at most tamely ramified above $p$. Then there exist finite sets $T$ of primes of $L$ such that $(\text{cl}_L^T)^- \otimes \mathbb{Z}_p$ is cohomologically trivial.

- Indeed, a slightly weaker condition on the primes above $p$ suffices.
Theorem

Let $L/K$ be a Galois CM-extension with Galois group $G$ and $p \neq 2$ a prime such that $L/K$ is at most tamely ramified above $p$. Then there exist finite sets $T$ of primes of $L$ such that $(\text{cl}_L^T)^- \otimes \mathbb{Z}_p$ is cohomologically trivial.

Indeed, a slightly weaker condition on the primes above $p$ suffices.

$(\text{cl}_L^T)^- \otimes \mathbb{Z}_p$ defines a class in $K_0(\mathbb{Z}_p G_-, \mathbb{Q}_p)$. 
Let $L/K$ be a Galois CM-extension with Galois group $G$ and $p \neq 2$ a prime, such that $L/K$ is at most tamely ramified above $p$. Then there exist finite sets $T$ of primes of $L$ such that the homomorphism

$$\chi \mapsto c_{S_\infty}(\tilde{\chi}) \prod_{p \in T(K)} \det(1 - \phi_\mathfrak{p}^{-1} N(p)| V_{\chi, \mathfrak{p}})$$

represents the class of $(c_{1_L}^T)^{-} \otimes \mathbb{Z}_p$ in $K_0(\mathbb{Z}_pG_-, \mathbb{Q}_p)$ if and only if the LRNC at $p$ holds on minus parts.
Theorem

Let $L/K$ be a Galois CM-extension with Galois group $G$ and $p \neq 2$ a prime, such that $L/K$ is at most tamely ramified above $p$. Then there exist finite sets $T$ of primes of $L$ such that the homomorphism

$$
\chi \mapsto c_{S_\infty}(\tilde{\chi}) \prod_{p \in T(K)} \det(1 - \phi_p^{-1}N(p)|V_{\chi,p}^l)
$$

represents the class of $(c_{1_L}^T)^{-} \otimes \mathbb{Z}_p$ in $K_0(\mathbb{Z}_p G_-, \mathbb{Q}_p)$ if and only if the LRNC at $p$ holds on minus parts.

- For the proof, we use the LRNC for a (small!) set $S$ of places which contains only totally decomposed primes.
Theorem

If in addition $G$ is abelian, the LRNC at $p$ holds on minus parts for almost all $p$. 
Abelian CM-extensions

Theorem

If in addition $G$ is abelian, the LRNC at $p$ holds on minus parts for almost all $p$.

- The main ingredient of the proof is the equivariant Iwasawa main conjecture.
- This conjecture is verified for abelian $G$ (A. Wiles 1990 for $p 
- The “descent” uses methods of A. Wiles in an extended version of C. Greither.
Proposition (Ritter/Weiss)

Let $L/K$ be a Galois extension of number fields and $p$ a prime. Assume that Stark’s conjecture holds. Then the Strong Stark Conjecture at $p$ holds if and only if the Strong Stark Conjecture holds for each cyclic intermediate extension of degree prime to $p$. 
Proposition (Ritter/Weiss)

Let $L/K$ be a Galois extension of number fields and $p$ a prime. Assume that Stark’s conjecture holds. Then the Strong Stark Conjecture at $p$ holds if and only if the Strong Stark Conjecture holds for each cyclic intermediate extension of degree prime to $p$.

The last Theorem + CM-version of the above proposition implies

Corollary

Let $L/K$ be any Galois CM-extension. Then the Strong Stark Conjecture at $p$ holds on minus parts for almost all primes $p$. 
Motivation

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Galois module structure

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Related conjectures
Let $L/K$ be an abelian CM-extension and $S$ and $T$ finite sets of primes of $L$ such that

- $S_{\text{ram}} \cup S_\infty \subset S$
- $T \neq \emptyset$
- $S \cap T = \emptyset$
- $\{\zeta \in \mu_L|\zeta \equiv 1 \mod \mathfrak{p} \ \forall \mathfrak{p} \in T\} = 1$
Strong Brumer-Stark I

Let $L/K$ be an abelian CM-extension and $S$ and $T$ finite sets of primes of $L$ such that

- $S_{\text{ram}} \cup S_\infty \subset S$
- $T \neq \emptyset$
- $S \cap T = \emptyset$
- $\{ \zeta \in \mu_L | \zeta \equiv 1 \mod \mathfrak{p} \ \forall \mathfrak{p} \in T \} = 1$

Define a **Stickelberger element**

$$\theta_T^S := \prod_{p \in T(K)} (1 - \phi_{\mathfrak{p}}^{-1} N(p))(\sum_{\chi \text{ irr.}} L_S(L/K, \check{\chi}, 0)e_\chi) \in \mathbb{Z}G$$

where $e_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})g$. 
Let $R = \mathbb{Z}G_- = \mathbb{Z}G/(1+j)$. Choose a resolution

$$R^a \xrightarrow{h} R^b \twoheadrightarrow (\text{cl}_L^T)^-$$

and define the Fitting ideal

$$\text{Fitt}_R((\text{cl}_L^T)^-) = \langle b \times b - \text{minors of } h \rangle_R$$
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Remark: $\text{Fitt}_R((\text{cl}_L^T)^-) \subset \text{Ann}_R((\text{cl}_L^T)^-)$.  

**Conjecture (Strong Brumer-Stark)**

$$\theta_S^T \in \text{Fitt}_R((\text{cl}_L^T)^-)$$
Overview

\[ L/K \] CM-extension, abelian

\[
\begin{align*}
\text{LRNC} & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \frac{\text{ETNC}}{\text{LRNC}}
Overview

$L/K$ CM-extension, abelian

- LRNC $\uparrow$ ETNC $\Rightarrow$ Strong Brumer-Stark $\downarrow$ Rubin-Stark $\Rightarrow$ Brumer-Stark
- Strong Stark $\Rightarrow$ Stark $\Rightarrow$ Brumer

- ETNC = equivariant Tamagawa Number Conjecture
- D. Burns 2007: ETNC $\Rightarrow$ Rubin-Stark
Overview

$L/K$ CM-extension, abelian

LRNC \uparrow \iff ETNC \downarrow \Rightarrow \text{Strong Brumer-Stark} \downarrow \Rightarrow \text{Rubin-Stark} \Rightarrow \text{Brumer-Stark}

\downarrow \Rightarrow \text{Strong Stark} \Rightarrow \text{Stark} \Rightarrow \text{Brumer}

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- C. Greither, M. Kurihara (preprint): wildly ramified counter-examples of Strong Brumer-Stark
Overview

$L/K$ CM-extension, abelian and tame

- LRNC \(\implies\) Strong Brumer-Stark
- ETNC \(\implies\) Rubin-Stark \(\implies\) Brumer-Stark
- Strong Stark \(\implies\) Stark \(\implies\) Brumer

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- Generalization to non-abelian $G$ assuming the validity of the equivariant Iwasawa main conjecture
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- On plus parts:
  - Stark’s Conjecture
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  - Does LRNC predict annihilators of the class group?
- On minus parts:
  - How to prove LRNC on minus parts for all primes $p \ (p \neq 2)$?
  - Find a new descent method