AN UNCONDITIONAL PROOF OF THE ABELIAN EQUIVARIANT IWASAWA MAIN CONJECTURE AND APPLICATIONS

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Abstract. Let \( p \) be an odd prime. We give an unconditional proof of the equivariant Iwasawa main conjecture for totally real fields for every admissible one-dimensional \( p \)-adic Lie extension whose Galois group has an abelian \( p \)-Sylow subgroup. Crucially, this result does not depend on the vanishing of any \( \mu \)-invariant. As applications, we deduce the Coates–Sinnott conjecture away from its 2-primary part and new cases of the equivariant Tamagawa number conjecture for Tate motives.

1. Introduction

Let \( p \) be an odd prime and let \( K \) be a totally real number field. An admissible \( p \)-adic Lie extension \( \mathcal{L} \) of \( K \) is a Galois extension \( \mathcal{L} \) of \( K \) such that (i) \( \mathcal{L}/K \) is unramified outside a finite set of primes of \( K \), (ii) \( \mathcal{L} \) is totally real, (iii) \( \mathcal{G} := \text{Gal}(\mathcal{L}/K) \) is a compact \( p \)-adic Lie group, and (iv) \( \mathcal{L} \) contains the cyclotomic \( \mathbb{Z}_p \)-extension of \( K \). The equivariant Iwasawa main conjecture (EIMC) for such an extension \( \mathcal{L}/K \) can be seen as a refinement and generalisation of the classical Iwasawa main conjecture for totally real fields proven by Wiles [Wil90]. Roughly speaking, it relates a certain Iwasawa module attached to \( \mathcal{L}/K \) to special values of Artin \( L \)-functions via \( p \)-adic \( L \)-functions. This relationship can be expressed as the existence of a certain element in an algebraic \( K \)-group; it is also conjectured that this element is unique.

Let \( S \) be a finite set of places of \( K \) containing all archimedean places and all places that ramify in \( \mathcal{L} \) (thus \( S \) necessarily contains all primes above \( p \)). Let \( M_S^{ab}(p) \) be the maximal abelian pro-\( p \)-extension of \( \mathcal{L} \) unramified outside \( S \) and set \( X_S = \text{Gal}(M_S^{ab}(p)/\mathcal{L}) \). The canonical short exact sequence

\[
1 \rightarrow X_S \rightarrow \text{Gal}(M_S^{ab}(p)/K) \rightarrow \mathcal{G} \rightarrow 1
\]

defines an action of \( \mathcal{G} \) on \( X_S \) in the usual way so that \( X_S \) becomes a module over the Iwasawa algebra \( \Lambda(\mathcal{G}) := \mathbb{Z}_p[\mathcal{G}] \). If \( \mathcal{G} \) contains no elements of order \( p \), then \( X_S \) is of finite projective dimension over \( \Lambda(\mathcal{G}) \), and so the EIMC can be stated in terms of \( X_S \). In general, however, \( X_S \) is not of finite projective dimension and so one has to replace \( X_S \) by a certain canonical complex \( C_S^* \) of \( \Lambda(\mathcal{G}) \)-modules which is perfect and whose only non-vanishing cohomology groups are isomorphic to \( X_S \) and \( \mathbb{Z}_p \), respectively.

There are several versions of the EIMC. The first is due to Ritter and Weiss and deals with the case of one-dimensional extensions [RW04], and was proven under the hypothesis that the \( \mu \)-invariant of \( X_S \) vanishes in a series of articles culminating in [RW11]. In their approach the complex \( C_S^* \) is obtained from the canonical group extension (1.1) by applying a certain ‘translation functor’ [RW02, §4A] which essentially transfers (1.1) into an arrow

\[
\mathcal{Y}_S \rightarrow \Lambda(\mathcal{G})
\]
with kernel $X_S$ and cokernel $\mathbb{Z}_p$. It can be shown that this arrow defines a complex with the required properties. The second version follows the framework of Coates, Fukaya, Kato, Sujatha and Venjakob [CFK+] and was proven by Kakde [Kak13], again assuming $\mu = 0$. This version is for arbitrary admissible extensions and Kakde’s proof uses a strategy of Burns and Kato to reduce to the one-dimensional case (see Burns [Bur15]). Here, the choice of complex appears to be different, but in the one-dimensional case both complexes are isomorphic in the derived category of $\Lambda(G)$-modules by a result of the second named author [Nic13, Theorem 2.4] (see also Venjakob [Ven13] for a thorough discussion of the relation of the work of Ritter and Weiss to that of Kakde.) As a consequence, it does not matter which of the two complexes we use. Finally, Greither and Popescu [GP15] formulated and proved another version of the EIMC, but they restricted their formulation to abelian one-dimensional extensions and the formulation itself requires a $\mu = 0$ hypothesis. In [Nic13], the second named author generalised this formulation (again assuming $\mu = 0$) to the non-abelian one-dimensional case. Moreover, he showed that the three formulations are in fact all equivalent in the situation that they make sense, that is, when the extension is one-dimensional and $\mu = 0$. In fact, the proof of this result shows that the choice of complex $C_S^\bullet$ is irrelevant when $\mu = 0$ (as long as it is perfect and has the prescribed cohomology). The idea is that the classical Iwasawa main conjecture for totally real fields proven by Wiles [Wil90] allows one to localise at the prime ideal generated by $p$, and in this setting all cohomology groups vanish if $\mu = 0$.

From a result of Ferrero and Washington [FW79], one can deduce that the $\mu = 0$ hypothesis holds whenever $\mathcal{L}/K$ is an admissible extension such that $\mathcal{L}$ is a pro-$p$ extension of a finite abelian extension of $\mathbb{Q}$, but unfortunately little is known beyond this case. In previous work [JN18], the present authors proved the EIMC unconditionally for an infinite class of one-dimensional admissible extensions for which the $\mu = 0$ hypothesis is not known to be true. However, such extensions must satisfy certain rather restrictive hypotheses, which, in particular, imply that the choice of the complex $C_S^\bullet$ is again irrelevant.

In the present article, we prove the EIMC (with uniqueness) in important cases without assuming any $\mu = 0$ hypothesis. The proof relies on the recent groundbreaking work of Dasgupta and Kakde [DK20] on the strong Brumer–Stark conjecture and a formulation of the EIMC given in the present authors’ article [JN19]. We emphasise that without the $\mu = 0$ hypothesis, the class of $C_S^\bullet$ in the derived category does indeed matter, and this is where our previous results will play a decisive role.

**Theorem 1.1.** Let $p$ be an odd prime and let $K$ be a totally real number field. Let $\mathcal{L}/K$ be an abelian admissible one-dimensional $p$-adic Lie extension. Then the EIMC with uniqueness holds for $\mathcal{L}/K$.

It is natural to ask whether one can deduce the EIMC for all admissible one-dimensional $p$-adic Lie extensions from Theorem 1.1 by generalising the approaches of Ritter and Weiss and of Kakde. The first step is to reduce to admissible subextensions with $p$-elementary Galois groups. In the aforementioned approaches, this step relied on the $\mu = 0$ hypothesis. By showing that certain products of maps are injective and exploiting the functorial properties of the EIMC, we obtain a similar result without any such hypothesis. We hence deduce the following generalisation of Theorem 1.1.

**Corollary 1.2.** Let $p$ be an odd prime and let $K$ be a totally real number field. Let $\mathcal{L}/K$ be an admissible one-dimensional $p$-adic Lie extension such that the $p$-Sylow subgroups of $\text{Gal}(\mathcal{L}/K)$ are abelian. Then the EIMC with uniqueness holds for $\mathcal{L}/K$. 
The further reduction steps of previous approaches do not generalise easily as they rely on the \( \mu = 0 \) hypothesis in a crucial way and hence presently there is no apparent way to deduce the EIMC for all admissible one-dimensional extensions without this hypothesis. Moreover, a serious obstacle to the case of admissible extensions of dimension greater than one is that in general a certain \( \mathfrak{M}_p(G) \)-conjecture is required to even formulate the EIMC in this situation, and that this is presently only known to hold under the \( \mu = 0 \) hypothesis (see [CK13, p. 5] and [CS12]).

We remark that if Leopoldt’s conjecture holds for \( K \) at \( p \) then every abelian admissible extension of \( K \) must be one-dimensional. Similarly, if Leopoldt’s conjecture holds for \( F \) at \( p \) for all finite totally real extensions \( F/K \) with \( [F : K] \) coprime to \( p \) then every admissible extension of \( K \) whose Galois group has abelian Sylow \( p \)-subgroups must be one-dimensional. Hence the hypothesis that the extensions considered in Theorem 1.1 and Corollary 1.2 are one-dimensional is not really restrictive. Moreover, the one-dimensional case of the EIMC often suffices for applications.

The equivariant Tamagawa number conjecture (ETNC) has been formulated by Burns and Flach [BF01] in vast generality. In the case of Tate motives it simply asserts that an associated canonical element in a relative algebraic \( G \)-group vanishes. Roughly speaking, this element relates leading terms of Artin \( L \)-functions to natural arithmetic invariants.

Let \( L/K \) be a finite Galois CM extension of number fields with Galois group \( G \). Hence \( L \) is a totally complex number field and complex conjugation induces a unique central automorphism in \( G \). Let \( r \) be a negative integer. In the case that the \( \mu = 0 \) hypothesis holds (for the cyclotomic \( \mathbb{Z}_p \)-extension of the maximal totally real subfield of \( L(\zeta_p) \), where \( \zeta_p \) denotes a primitive \( p \)th root of unity), it is known by independent work of Burns [Bur15] and of the second named author [Nic13] that the EIMC implies the plus (resp. minus) \( p \)-part of the ETNC for the pair \( (h^0(\text{Spec}(L))(r), \mathbb{Z}[G]) \) if \( r \) is odd (resp. even). In both approaches to this result, \( \mu = 0 \) was mainly assumed to ensure the validity of the EIMC. Thus at first sight, Theorem 1.3 below appears to be a direct consequence of our results on the EIMC above. However, Burns’ descent argument relies on the formalism developed by Burns and Venjakob in [BV11]. For this, the cohomology of a certain complex at infinite \( \mathbb{Z} \)-functions to natural arithmetic invariants. Hence the hypothesis that the extensions considered in Theorem 1.1 and Corollary 1.2 are one-dimensional is not really restrictive. Moreover, the one-dimensional case of the EIMC often suffices for applications.

Theorem 1.3. Let \( p \) be an odd prime. Let \( L/K \) be a finite Galois CM extension of number fields with Galois group \( G \). Suppose that \( G \) has an abelian Sylow \( p \)-subgroup. Then for each negative odd (resp. even) integer \( r \) the plus (resp. minus) \( p \)-part of the ETNC for the pair \( (h^0(\text{Spec}(L))(r), \mathbb{Z}[G]) \) holds.

Now assume that \( L/K \) is a finite abelian extension of number fields. Let \( S \) be a finite set of places of \( K \) that contains all archimedean places and all places that ramify in \( L \). We write \( \mathcal{O}_{L,S} \) for the ring of \( S(L) \)-integers in \( L \), where \( S(L) \) denotes the set of places of \( L \) that lie above a place in \( S \). For an integer \( n \geq 0 \) we let \( K_n(\mathcal{O}_{L,S}) \) denote the Quillen \( K \)-theory of \( \mathcal{O}_{L,S} \). Using \( L \)-values at negative integers \( r \) one can define Stickelberger elements \( \theta_S(r) \) in the rational group ring \( \mathbb{Q}[G] \). If we write \( K_{1-2r}(\mathcal{O}_L)_{\text{tors}} \) for the torsion subgroup of \( K_{1-2r}(\mathcal{O}_L) \), then one knows from independent work of Deligne and Ribet
Coates and Sinnott [CS74] formulated the following analogue of Brumer’s conjecture for higher $K$-groups.

**Conjecture 1.4** (Coates–Sinnott). Let $L/K$ be a finite abelian extension of number fields with Galois group $G$. Let $r$ be a negative integer and let $S$ be a finite set of places of $K$ that contains all archimedean places and all places that ramify in $L$. Then one has

$$\text{Ann}_{\mathbb{Z}[G]}(K_{1-2r}(\mathcal{O}_L)_{\text{tors}})\theta_S(r) \subseteq \mathbb{Z}[G].$$

Let $p$ be an odd prime and suppose in addition that $S$ contains all $p$-adic places of $K$. For any negative integer $r$ and $i = 0, 1$ Soulé [Sou79] has constructed canonical $G$-equivariant $p$-adic Chern class maps

$$Z_p \otimes_{\mathbb{Z}} K_{i-2r}(\mathcal{O}_{L,S}) \rightarrow H^{2-i}_{\text{ét}}(\text{Spec}(\mathcal{O}_{L,S}), \mathbb{Z}_p(1-r)).$$

Soulé proved surjectivity and by the norm residue isomorphism theorem [Wei09] (formerly known as the Quillen–Lichtenbaum Conjecture) these maps are actually isomorphisms.

This allows us to work with an étale cohomological version of the conjecture. For a variant of the latter it has been shown in [GP15, §6] that it suffices to consider abelian CM extensions. We therefore obtain the following consequence of Theorem 1.3.

**Theorem 1.5.** The Coates–Sinnott conjecture holds away from its 2-primary part.

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**Notation and conventions.** All rings are assumed to have an identity element and all modules are assumed to be left modules unless otherwise stated. We fix the following notation:

- $R^*$: the group of units of a ring $R$
- $\zeta(R)$: the centre of a ring $R$
- $\text{Ann}_R(M)$: the annihilator of the $R$-module $M$
- $\text{Fitt}_R(M)$: the (initial) Fitting ideal of a finitely presented module $M$
- $M_n(R)$: the set of all $n \times n$ matrices with entries in a ring $R$
- $\text{Quot}(R)$: the field of fractions of the integral domain $R$
- $\zeta_n$: a primitive $n$th root of unity
- $K_{\infty}$: the cyclotomic $\mathbb{Z}_p$-extension of the number field $K$
- $\text{cl}_K$: the class group of a number field $K$
- $K^+$: an algebraic closure of a field $K$
- $K'$: the maximal totally real subfield of a field $K$ embeddable into $\mathbb{C}$
- $\text{Irr}_F(G)$: the set of $F$-irreducible characters of the (pro)-finite group $G$ (with open kernel) where $F$ is a field of characteristic 0
- $\bar{\chi}$: the character contragredient to $\chi$
- $\text{Re}(s)$: the real part of the complex number $s$
2. The Brumer–Stark conjecture

2.1. Ray class groups. Let $L/K$ be a finite Galois extension of number fields and let $G = \text{Gal}(L/K)$. For each place $v$ of $K$ we fix a place $w$ of $L$ above $v$ and write $G_w$ and $I_w$ for the decomposition group and the inertia subgroup of $G$ at $w$, respectively. When $w$ is a finite place, we choose a lift $\sigma_w \in G_w$ of the Frobenius automorphism at $w$ and write $\mathfrak{P}_w$ for the associated prime ideal in $L$.

For any set $S$ of places of $K$, we write $S(L)$ for the set of places of $L$ which lie above those in $S$. Now let $S$ be a finite set of places of $K$ containing the set $S_{\infty} = S_{\infty}(K)$ of archimedean places and let $T$ be a second finite set of places of $K$ such that $S \cap T = \emptyset$. We write $\text{cl}^T_L$ for the ray class group of $L$ associated to the modulus $\mathfrak{M}_L^T := \prod_{w \in T(L)} \mathfrak{P}_w$ and $\mathcal{O}_{L,S}$ for the ring of $S(L)$-integers in $L$. If $T$ is empty we abbreviate $\text{cl}^T_L$ to $\text{cl}_L$. Let $\mathcal{O}_L := \mathcal{O}_{L,S_{\infty}}$ be the ring of integers in $L$. We denote the group $\mathcal{O}_L^\times$ of units in $L$ by $E_L$ and define $E_L^T := \{x \in E_L : x \equiv 1 \text{ mod } \mathfrak{M}_L^T\}$. All these modules are equipped with a natural $G$-action.

2.2. Equivariant Artin $L$-functions and values. Let $S$ be a finite set of places of $K$ containing the infinite places $S_{\infty}$. For a finite place $v$ of $K$ we denote the cardinality of its residue field by $N_v$. Let $\text{Irr}_C(G)$ denote the set of complex irreducible characters of $G$. For $\chi \in \text{Irr}_C(G)$ let $V_\chi$ be a left $C[G]$-module with character $\chi$. The $S$-truncated Artin $L$-function $L_S(s, \chi)$ is defined as the meromorphic extension to the whole complex plane of the holomorphic function given by the Euler product

$$L_S(s, \chi) = \prod_{v \notin S} \det(1 - (Nv)^{-s}\sigma_w \mid V_\chi^w)^{-1}, \quad \text{Re}(s) > 1.$$ 

The primitive central idempotents of $C[G]$ attached to elements of $\text{Irr}_C(G)$ form a $C$-basis of $\zeta(C[G])$ and thus there is a canonical isomorphism $\zeta(C[G]) \cong \prod_{\chi \in \text{Irr}_C(G)} C$. The equivariant $S$-truncated Artin $L$-function is defined to be the meromorphic $\zeta(C[G])$-valued function

$$L_S(s) := (L_S(s, \chi))_{\chi \in \text{Irr}_C(G)}.$$ 

Now suppose that $T$ is a second finite set of places of $K$ such that $S \cap T = \emptyset$. Then we define

$$\delta_T(s, \chi) := \prod_{v \in T} \det(1 - (Nv)^{-1}\sigma_w^{-1} \mid V_\chi^w)^{-1} \quad \text{and} \quad \delta_T(s) := (\delta_T(s, \chi))_{\chi \in \text{Irr}_C(G)}.$$ 

The $(S, T)$-modified $G$-equivariant Artin $L$-function is defined to be

$$\Theta_{S,T}(s) := \delta_T(s) \cdot L_S(s)^\#,$$

where $\# : C[G] \to C[G]$ denotes the anti-involution induced by $g \mapsto g^{-1}$ for $g \in G$. Note that $L_S(s)^\# = (L_S(s, \tilde{\chi}))_{\chi \in \text{Irr}_C(G)}$ where $\tilde{\chi}$ denotes the character contragredient to $\chi$. Evaluating $\Theta_{S,T}(s)$ at $s = 0$ gives an $(S, T)$-modified Stickelberger element

$$\theta_T^S(L/K) = \theta_S^T := \Theta_{S,T}(0) \in \zeta(C[G]).$$ 

Note that a priori we only have $\theta_T^S \in \zeta(C[G])$, but by a result of Siegel [Sie70] we know that $\theta_T^S$ in fact belongs to $\zeta(Q[G])$. If $T$ is empty, we abbreviate $\theta_T^S$ to $\theta_S$. 


2.3. The Brumer and Brumer–Stark conjectures for abelian extensions. We now specialise to the case in which $L/K$ is an abelian CM extension of number fields. In other words, $K$ is totally real and $L$ is a finite abelian extension of $K$ that is a CM field. Let $\mu_L$ and $\text{cl}_L$ denote the roots of unity and the class group of $L$, respectively.

Let $S_{\text{ram}} = S_{\text{ram}}(L/K)$ be the set of all places of $K$ that ramify in $L/K$. It was shown independently by Pi. Cassou-Noguès [CN79] and by Deligne and Ribet [DR80] that

$$\text{Ann}_{\mathbb{Z}[G]}(\mu_L)\theta_S \subseteq \mathbb{Z}[G].$$

Brumer’s conjecture simply asserts that $\text{Ann}_{\mathbb{Z}[G]}(\mu_L)\theta_S$ annihilates $\text{cl}_L$ and in the case $K = \mathbb{Q}$ this is Stickelberger’s theorem [Sti90].

**Hypothesis.** Let $S$ and $T$ be finite sets of places of $K$. We say that Hyp$(S,T)$ is satisfied if (i) $S_{\text{ram}} \cup S_\infty \subseteq S$, (ii) $S \cap T = \emptyset$, and (iii) $E_L^T$ is torsionfree.

**Remark 2.1.** Condition (iii) means that there are no non-trivial roots of unity of $L$ congruent to 1 modulo all primes in $T(L)$. In particular, this will be satisfied if $T$ contains primes of two different residue characteristics or at least one prime of sufficiently large norm.

If $S$ and $T$ are finite sets of places of $K$ satisfying Hyp$(S,T)$ then (2.1) implies that $\theta_S^T \in \mathbb{Z}[G]$. Moreover, given a finite set $S$ of places of $K$ such that $S_{\text{ram}} \cup S_\infty \subseteq S$, Brumer’s conjecture for $S$ holds if and only if $\theta_S^T \in \text{Ann}_{\mathbb{Z}[G]}(\text{cl}_L)$ for every finite set of places $T$ of $L$ such that Hyp$(S,T)$ is satisfied. (See [Nic19a, Corollary 2.9]). The following strengthening of Brumer’s conjecture was stated by Tate and is known as the Brumer–Stark conjecture.

**Conjecture 2.2.** For every pair $S,T$ of finite sets of places of $K$ satisfying Hyp$(S,T)$ we have $\theta_S^T \in \text{Ann}_{\mathbb{Z}[G]}(\text{cl}_L^T)$.

In fact, as explained in [DK20, §1], Conjecture 2.2 is slightly different from the actual statement proposed by Tate [Tat84, Conjecture IV.6.2], but it is the former that will be the most convenient for our purposes. We also note that Conjecture 2.2 decomposes into local conjectures at each prime $p$ after replacing $\text{cl}_L^T$ by $\mathbb{Z}_p \otimes_{\mathbb{Z}} \text{cl}_L^T$.

For generalisations of the Brumer–Stark conjecture to not necessarily abelian extensions we refer the interested reader to the survey article [Nic19a].

2.4. The strong Brumer–Stark conjecture for abelian extensions. If $M$ is a finitely presented module over a commutative ring $R$, we denote the (initial) Fitting ideal of $M$ over $R$ by $\text{Fitt}_R(M)$. For an abstract abelian group $A$ we write $A^\vee$ for the Pontryagin dual $\text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. This induces an equivalence between the categories of abelian profinite groups and discrete abelian torsion groups (see [NSW08, Theorem 1.1.11] and the discussion thereafter).

For a finitely generated $\mathbb{Z}_p[G]$-module $M$, we have $M^\vee = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p)$, and this is endowed with the contragredient $G$-action $(gf)(m) = f(g^{-1}m)$ for $f \in M^\vee$, $g \in G$ and $m \in M$. Let $j$ denote the unique complex conjugation in $G$. For a $G$-module $M$ we write $M^+$ and $M^-$ for the submodules of $M$ upon which $j$ acts as 1 and $-1$, respectively. In particular, we shall be interested in $(\mathbb{Z}_p \otimes_{\mathbb{Z}} \text{cl}_L^T)^+$ for odd primes $p$; we will abbreviate this module to $A_L^{T+}$ when $p$ is clear from context; if $T$ is empty we further abbreviate this to $A_L$. Note that $A_L^{T+}$ and $(A_L^{T+})^\vee$ are modules of finite cardinality over the ring $\mathbb{Z}_p[G]_- := \mathbb{Z}_p[G]/(1 + j)$. The following groundbreaking result was recently proven by Dasgupta and Kakde [DK20, Corollary 3.7].
Theorem 2.3. Let \( S, T \) be finite sets of places of \( K \) such that \( \text{Hyp}(S,T) \) is satisfied. Then for every odd prime \( p \) we have \((\theta^T_p)^\# \in \text{Fitt}_{\mathbb{Z}/p}\mathfrak{g}((A^T_L)^\vee)\).

Theorem 2.3 can be seen as a refinement of the ‘\( p \)-part’ of Conjecture 2.2 (with \( p \) odd), once we observe that: (i) the Fitting ideal of a module is contained in its annihilator; (ii) \( \text{Ann}_{\mathbb{Z}/p}\mathfrak{g}((M)^\#) = \text{Ann}_{\mathbb{Z}/p}\mathfrak{g}((M')^\#) \) for every \( \mathbb{Z}/p\mathfrak{g} \)-module \( M \) of finite cardinality; and (iii) \( j \) acts as \(-1\) on \( \theta^T_p \), so the element \( \theta^T_p \) annihilates a \( \mathbb{Z}/p\mathfrak{g} \)-module \( M \) if and only if it annihilates \( M^- \).

Remark 2.4. Greither and Kurihara [GK08] have given counterexamples to the ‘dual’ version of Theorem 2.3, which asserts that \( \theta^T_p \in \text{Fitt}_{\mathbb{Z}/p}\mathfrak{g}((A^T_L)^\vee) \) under the same hypotheses. They have also given counterexamples to the assertion \( \theta^T_p \in \text{Fitt}_{\mathbb{Z}/p}\mathfrak{g}((A^T_L)^\vee) \) [GK15, §0.1]; these do not contradict Theorem 2.3 thanks to Hyp(\( S,T \)) for the ‘dual’ version of Theorem 2.3, which asserts that \( \theta^T_p \in \text{Fitt}_{\mathbb{Z}/p}\mathfrak{g}((A^T_L)^\vee) \) under the same hypotheses.

3. Algebraic \( K \)-theory and Iwasawa algebras

3.1. Algebraic \( K \)-theory. Let \( \Lambda \) be a left Noetherian ring and let \( \text{PMod}(\Lambda) \) denote the category of finitely generated projective (left) \( \Lambda \)-modules. We write \( K_0(\Lambda) \) for the Grothendieck group of \( \text{PMod}(\Lambda) \) (see [CR87, §38]) and \( K_1(\Lambda) \) for the Whitehead group (see [CR87, §40]). Let \( K_0(\Lambda, \Lambda') \) denote the relative algebraic \( K \)-group associated to a ring homomorphism \( \Lambda \hookrightarrow \Lambda' \). We recall that \( K_0(\Lambda, \Lambda') \) is an abelian group with generators \([X, Y]\) where \( X \) and \( Y \) are finitely generated projective \( \Lambda \)-modules and \( g : \Lambda' \otimes_{\Lambda} X \to \Lambda' \otimes_{\Lambda} Y \) is an isomorphism of \( \Lambda' \)-modules; for a full description in terms of generators and relations, we refer the reader to [Swa68, p. 215]. Moreover, there is a long exact sequence of relative \( K \)-theory (see [Swa68, Chapter 15])

\[
K_1(\Lambda) \to K_1(\Lambda') \to K_0(\Lambda, \Lambda') \to K_0(\Lambda) \to K_0(\Lambda').
\]

3.2. Algebraic \( K \)-theory for orders in semisimple algebras. Let \( R \) be a noetherian integral domain. Let \( A \) be a finite-dimensional semisimple quotient \( \text{Quot}(R) \)-algebra and let \( \mathfrak{A} \) be an \( R \)-order in \( A \). The reduced norm map \( \text{nr} = \text{nr}_A : A \to \zeta(A) \) is defined componentwise on the Wedderburn decomposition of \( A \) and extends to matrix rings over \( A \) (see [CR81, §7D]); thus it induces a map \( K_1(A) \to \zeta(A)^{X} \), which we also denote by \( \text{nr} \).

Let \( \mathcal{C}^b(\text{PMod}(\mathfrak{A})) \) be the category of bounded chain complexes of finitely generated projective \( \mathfrak{A} \)-modules. Then \( K_0(\mathfrak{A}, A) \) identifies with the Grothendieck group whose generators are \([C^\bullet] \), where \( C^\bullet \) is an object of the category \( \mathcal{C}^b(\text{PMod}(\mathfrak{A})) \) of bounded complexes of finitely generated projective \( \mathfrak{A} \)-modules whose cohomology modules are \( R \)-torsion, and the relations are as follows: \([C^\bullet] = 0 \) if \( C^\bullet \) is acyclic, and \([C^2] = [C^1] + [C^3] \) for every short exact sequence

\[
0 \to C^\bullet_1 \to C^\bullet_2 \to C^\bullet_3 \to 0
\]

in \( \mathcal{C}^b(\text{PMod}(\mathfrak{A})) \) (see [Wei13, Chapter 2] or [Suj13, §2], for example). If \( n \) is an integer, we write \( [C^\bullet[n]] \) for the \( n \)-shifted chain complex and note that we have an equality \([C^\bullet[n]] = (-1)^n[C^\bullet] \) in \( K_0(\mathfrak{A}, A) \).

Let \( \mathcal{D}(\mathfrak{A}) \) be the derived category of \( \mathfrak{A} \)-modules. A complex of \( \mathfrak{A} \)-modules is said to be perfect if it is isomorphic in \( \mathcal{D}(\mathfrak{A}) \) to an element of \( \mathcal{C}^b(\text{PMod}(\mathfrak{A})) \). We denote the full triangulated subcategory of \( \mathcal{D}(\mathfrak{A}) \) comprising perfect complexes by \( \mathcal{D}^\text{perf}(\mathfrak{A}) \), and the full triangulated subcategory comprising perfect complexes whose cohomology modules are \( R \)-torsion by \( \mathcal{D}^\text{perf}_\text{tor}(\mathfrak{A}) \). Then any object of \( \mathcal{D}^\text{perf}_\text{tor}(\mathfrak{A}) \) defines an element in \( K_0(\mathfrak{A}, A) \).
3.3. Iwasawa algebras of one-dimensional compact $p$-adic Lie groups. Let $p$ be a prime and let $\mathcal{G}$ be a one-dimensional compact $p$-adic Lie group. In other words, $\mathcal{G}$ is a profinite group containing a finite normal subgroup $H$ such that $\bar{\Gamma} := \mathcal{G}/H$ is a pro-$p$-group isomorphic to $\mathbb{Z}_p$. The argument given in [RW04, §1] shows that the short exact sequence

$$1 \rightarrow H \rightarrow \mathcal{G} \rightarrow \bar{\Gamma} \rightarrow 1$$

splits. Thus we obtain a semidirect product $\mathcal{G} = H \rtimes \Gamma$ where $\Gamma \leq \mathcal{G}$ and $\Gamma \simeq \bar{\Gamma} \simeq \mathbb{Z}_p$. Note that the image under the natural projection map $\mathcal{G} \rightarrow \bar{\Gamma}$ of any element of $\mathcal{G}$ of finite order is also of finite order and hence must be trivial. Thus $H$ is equal to the subset of $\mathcal{G}$ of elements of finite order. Therefore $H$ and $\bar{\Gamma}$ are uniquely determined by $\mathcal{G}$, though the choice of $\Gamma$ need not be.

The Iwasawa algebra of $\mathcal{G}$ is $\Lambda(\mathcal{G}) := \mathbb{Z}_p[[\mathcal{G}]] = \lim \mathbb{Z}_p[\mathcal{G}/\mathcal{N}]$, where the inverse limit is taken over all open normal subgroups $\mathcal{N}$ of $\mathcal{G}$. If $\mathcal{F}$ is a finite field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O} = \mathcal{O}_F$, we put $\Lambda^\mathcal{O}(\mathcal{G}) := \mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda(\mathcal{G}) = \mathcal{O}[\mathcal{G}]$. We fix a topological generator $\gamma$ of $\Gamma$. Let $\mathcal{F} := \Gamma \bmod H$ and note that this a topological generator of $\bar{\Gamma}$. Since any homomorphism $\Gamma \rightarrow \text{Aut}(H)$ must have open kernel, we may choose a natural number $n$ such that $\gamma^n$ is central in $\mathcal{G}$; we fix such an $n$. As $\Gamma_n := \Gamma^n \simeq \mathbb{Z}_p$, there is a ring isomorphism $R := \mathcal{O}[\Gamma_n] \simeq \mathcal{O}[T]$ induced by $\gamma^n \mapsto 1 + T$ where $\mathcal{O}[T]$ denotes the power series ring in one variable over $\mathcal{O}$. If we view $\Lambda^\mathcal{O}(\mathcal{G})$ as an $R$-module (or indeed as a left $R[\mathcal{H}]$-module), there is a decomposition

$$\Lambda^\mathcal{O}(\mathcal{G}) = \bigoplus_{i=0}^{p^n-1} R[\mathcal{H}]\gamma^i. \tag{3.3}$$

Hence $\Lambda^\mathcal{O}(\mathcal{G})$ is finitely generated as an $R$-module and is an $R$-order in the separable $\text{Quot}(R)$-algebra $\mathcal{Q}^\mathcal{F}(\mathcal{G})$, the total ring of fractions of $\Lambda^\mathcal{O}(\mathcal{G})$, obtained from $\Lambda^\mathcal{O}(\mathcal{G})$ by adjoining inverses of all central regular elements. Note that $\mathcal{Q}^\mathcal{F}(\mathcal{G}) = \text{Quot}(R) \otimes_R \Lambda^\mathcal{O}(\mathcal{G})$ and that by [RW04, Lemma 1] we have $\mathcal{Q}^\mathcal{F}(\mathcal{G}) = F \otimes_{\mathbb{Q}_p} \mathcal{Q}(\mathcal{G})$, where $\mathcal{Q}(\mathcal{G}) := \mathcal{Q}^\mathcal{F}(\mathcal{G})$.

3.4. Algebraic $K$-theory for Iwasawa algebras. We now specialise §3.2 to the situation of §3.3. Let $p$ be a prime and let $\mathcal{G} = H \rtimes \Gamma$ be a one-dimensional compact $p$-adic Lie group. Let $\Gamma_n$ be an open subgroup of $\Gamma$ that is central in $\mathcal{G}$ and let $F$ be a finite field extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O} = \mathcal{O}_F$. Let $A = \mathcal{Q}^\mathcal{F}(\mathcal{G})$, let $\mathfrak{K} = \Lambda^\mathcal{O}(\mathcal{G}) = \mathcal{O}[\mathcal{G}]$ and let $R = \mathcal{O}[\Gamma_n]$. Then since $\mathcal{G}$ is one-dimensional [Wit13, Corollary 3.8] shows that the map $\partial_1$ in (3.1) is surjective; thus we obtain an exact sequence

$$K_1(\Lambda^\mathcal{O}(\mathcal{G})) \rightarrow K_1(\mathcal{Q}^\mathcal{F}(\mathcal{G})) \stackrel{\partial_1}{\rightarrow} K_0(\Lambda^\mathcal{O}(\mathcal{G}), \mathcal{Q}^\mathcal{F}(\mathcal{G})) \rightarrow 0. \tag{3.4}$$

Let $N$ be a finite normal subgroup of $\mathcal{G}$. Then there is a natural commutative diagram

$$\begin{array}{ccc}
\Lambda^\mathcal{O}(\mathcal{G}) & \longrightarrow & \mathcal{Q}^\mathcal{F}(\mathcal{G}) \\
\downarrow & & \downarrow \\
\Lambda^\mathcal{O}(\mathcal{G}/N) & \longrightarrow & \mathcal{Q}^\mathcal{F}(\mathcal{G}/N),
\end{array}$$
where the vertical arrows are the natural projections. This diagram and its vertical arrows induce canonical maps quot_{G/N}^G to give a commutative diagram

\[
\begin{align*}
K_1(\Lambda^O(G)) & \longrightarrow K_1(Q^F(G)) \longrightarrow K_0(\Lambda^O(G), Q^F(G)) \longrightarrow 0 \\
\text{quot}_{G/N}^G & \downarrow \quad \text{quot}_{G/N}^G \quad \text{quot}_{G/N}^G \\
K_1(\Lambda^O(G/N)) & \longrightarrow K_1(Q^F(G/N)) \longrightarrow K_0(\Lambda^O(G/N), Q^F(G/N)) \longrightarrow 0.
\end{align*}
\]

Let \( H \) be an open subgroup of \( G \). Then there is a natural commutative diagram of scalar extensions

\[
\begin{align*}
\Lambda^O(H) & \longrightarrow Q^F(H) \\
\Lambda^O(G) & \longrightarrow Q^F(G).
\end{align*}
\]

This diagram and its vertical arrows induce canonical maps res_H^G to give a commutative diagram

\[
\begin{align*}
K_1(\Lambda^O(G)) & \longrightarrow K_1(Q^F(G)) \longrightarrow K_0(\Lambda^O(G), Q^F(G)) \longrightarrow 0 \\
\text{res}_H^G & \downarrow \quad \text{res}_H^G \quad \text{res}_H^G \\
K_1(\Lambda^O(H)) & \longrightarrow K_1(Q^F(H)) \longrightarrow K_0(\Lambda^O(H), Q^F(H)) \longrightarrow 0.
\end{align*}
\]

### 3.5. Characters and central primitive idempotents

Fix a character \( \chi \in \text{Irr}_{Q_p}^G(G) \) (i.e. an irreducible \( Q_p \)-valued character of \( G \) with open kernel) and let \( \eta \) be an irreducible constituent of \( \text{res}_H^G \chi \). Then \( G \) acts on \( \eta \) as \( \eta^g(h) = \eta(g^{-1}hg) \) for \( g \in G, h \in H \), and following [RW04, §2] we set

\[
\text{St}(\eta) := \{ g \in G : \eta^g = \eta \}, \quad e(\eta) := \frac{\eta(1)}{|H|} \sum_{h \in H} \eta(h^{-1}h), \quad e_\chi := \sum_{\eta \in \text{res}_H^G \chi} e(\eta).
\]

By [RW04, Corollary to Proposition 6] \( e_\chi \) is a primitive central idempotent of \( Q^c(G) := Q_p \otimes_{Q_p} Q(G) \). In fact, every primitive central idempotent of \( Q^c(G) \) is of this form and \( e_\chi = e_\chi' \) if and only if \( \chi = \chi' \otimes \rho \) for some character \( \rho \) of \( G \) of type \( W \) (i.e. \( \text{res}_H^G \rho = 1 \)). Let \( w_\chi = [G : \text{St}(\eta)] \) and note that this is a power of \( p \) since \( H \) is a subgroup of \( \text{St}(\theta) \).

Let \( E/Q_p \) be a finite extension over which both characters \( \chi \) and \( \eta \) have realisations. Let \( V_\chi \) denote a realisation of \( \chi \) over \( E \). By [RW04, Propositions 5 and 6] and [JN18, Lemma 3.1], there exists a unique element \( \gamma_\chi \in \zeta(Q^E(G)e_\chi) \) such that \( \gamma_\chi \) acts trivially on \( V_\chi \) and \( \gamma_\chi = g_\chi c_\chi \) where \( g_\chi \in G \) with \( (g_\chi \mod H) = \overline{\eta^{w_\chi}} \) and with \( c_\chi \in (E[H]e_\chi)^\times \). Moreover, \( \gamma_\chi \) generates a pro-cyclic \( p \)-subgroup \( \Gamma_\chi \) of \( Q^E(G)e_\chi \) and induces an isomorphism

\[
Q^E(\Gamma_\chi) \cong \zeta(Q^E(G)e_\chi).
\]

### 3.6. Determinants and reduced norms

Following [RW04, Proposition 6], we define a map

\[
j_\chi : \zeta(Q^E(G)) \rightarrow \zeta(Q^E(G)e_\chi) \cong Q^E(\Gamma_\chi) \rightarrow Q^F(\Gamma),
\]

where the last arrow is induced by mapping \( \gamma_\chi \) to \( \overline{\eta^{w_\chi}} \). It follows from loc. cit. that \( j_\chi \) is independent of the choice of \( \overline{\eta} \) and that for every matrix \( \Theta \in M_n(Q^E(G)) \) we have

\[
j_\chi(\text{nr}(\Theta)) = \det_{Q^E(\Gamma)}(\Theta \mid \text{Hom}_{E[H]}(V_\chi, Q^E(G)^n)).
\]
Here, $\Theta$ acts on $f \in \text{Hom}_{E[G]}(V_\chi, Q^c(G)^n)$ via right multiplication, and $\gamma$ acts on the left via $(\gamma f)(v) = \gamma \cdot f(\gamma^{-1} v)$ for all $v \in V_\chi$ which is easily seen to be independent of the choice of $\gamma$. Let $F/\mathbb{Q}_p$ be a finite extension. By enlarging $E$ if necessary, we may assume that $F$ is a subfield of $E$. Then the map

\[
\text{Det}(\chi) : K_1(Q^c(G)) \to Q^c(\Gamma)^\times
\]

such that we obtain a commutative triangle

\[
\begin{array}{ccc}
K_1(Q^c(G)) & \to & Q^c(\Gamma)^\times \\
\text{nr} & \downarrow & \approx \\
\zeta(Q^c(G))^\times & \to & \text{Hom}_{G_p}^*(R_p(G), Q^c(\Gamma)^\times)
\end{array}
\]

where $P$ is a projective $Q^c(G)$-module and $\alpha$ a $Q^c(G)$-automorphism of $P$, is just $j_\chi \circ \text{nr}$ (see [RW04, §3, p. 558] for more details). If $\rho$ is a character of $G$ of type $W$ (i.e. $\text{res}_H^G \rho = 1$) then we denote by $\rho^\#$ the automorphism of the field $Q^c(\Gamma)$ induced by $\rho^\#(\gamma) = \rho(\gamma)\gamma$. Moreover, we denote the additive group generated by all $Q^c_p$-valued characters of $G$ with open kernel by $R_p(G)$; finally, $\text{Hom}_{G_p}^*(R_p(G), Q^c(\Gamma)^\times)$ is the group of all homomorphisms $f : R_p(G) \to Q^c(\Gamma)^\times$ satisfying

\[
f(\chi \otimes \rho) = \rho^\#(f(\chi)) \quad \text{for all characters } \rho \text{ of type } W \text{ and } \quad f(\sigma \chi) = \sigma(f(\chi)) \quad \text{for all Galois automorphisms } \sigma \in G_F.
\]

By [RW04, Theorem 7] (take $G_F$-invariants as in [RW04, Proof of Theorem 8]) we have an isomorphism

\[
\zeta(Q^c(G))^\times \cong \text{Hom}_{G_p}^*(R_p(G), Q^c(\Gamma)^\times)
\]

\[
x \mapsto [\chi \mapsto j_\chi(x)].
\]

As $\text{Det}(\chi) = \chi(\gamma)$ is just the composite map $j_\chi \circ \text{nr}$, the map $\Theta \mapsto [\chi \mapsto \text{Det}(\Theta)(\chi)]$ defines a homomorphism

\[
\text{Det} : K_1(Q^c(G)) \to \text{Hom}_{G_p}^*(R_p(G), Q^c(\Gamma)^\times)
\]

such that we obtain a commutative triangle

\[
\begin{array}{ccc}
K_1(Q^c(G)) & \to & \text{Hom}_{G_p}^*(R_p(G), Q^c(\Gamma)^\times) \\
\text{Det} & \downarrow & \approx \\
\zeta(Q^c(G))^\times & \to & \text{Hom}_{G_p}^*(R_p(G), Q^c(\Gamma)^\times)
\end{array}
\]

Let $N$ be a finite normal subgroup of $G$. Following [RW04, §3], we define a map

\[
\text{quot}_{G/N}^G : \text{Hom}_{G_p}^*(R_p(G), Q^c(\Gamma)^\times) \to \text{Hom}_{G_p}^*(R_p(G/N), Q^c(\Gamma)^\times),
\]

by $(\text{quot}_{G/N}^G f)(\chi) = f(\text{inf}_{G/N} G \chi)$ for $f \in \text{Hom}_{G_p}^*(R_p(G), Q^c(\Gamma)^\times)$ and $\chi \in R_p(G/N)$.

Let $\mathcal{H}$ be an open subgroup of $G$. As explained in §3.3, there exists a unique finite normal subgroup $H'$ of $\mathcal{H}$ such that $\Gamma_{\mathcal{H}} := \mathcal{H}/H'$ is a pro-$p$-group isomorphic to $\mathbb{Z}_p$. Moreover, there is a canonical embedding $\iota_{\mathcal{H}} : \Gamma_{\mathcal{H}} \to \Gamma$ defined as follows: given any element $x \in \Gamma_{\mathcal{H}}$, let $y \in \mathcal{H}$ be any lift and define $\iota_{\mathcal{H}}(x)$ to be the image of $y$ under the composition of canonical maps $\mathcal{H} \hookrightarrow G \to \Gamma$. It is straightforward to check that this map is well defined. Again following [RW04, §3], we define a map

\[
\text{res}_{\mathcal{H}}^G : \text{Hom}_{G_p}^*(R_p(G), Q^c(\Gamma_{\mathcal{H}})^\times) \to \text{Hom}_{G_p}^*(R_p(\mathcal{H}), Q^c(\Gamma_{\mathcal{H}})^\times),
\]

by $(\text{res}_{\mathcal{H}}^G f)(\chi') = f(\text{ind}_{\mathcal{H}'}^G \chi')$ for $f \in \text{Hom}_{G_p}^*(R_p(G), Q^c(\Gamma_{\mathcal{H}})^\times)$ and $\chi' \in R_p(\mathcal{H})$. Here we view $Q^c(\Gamma_{\mathcal{H}})$ as a subfield of $Q^c(\Gamma)$ via the embedding $\iota_{\mathcal{H}} : \Gamma_{\mathcal{H}} \hookrightarrow \Gamma$. 
Via diagram (3.8) the maps just defined induce canonical group homomorphisms
\[
\text{quot}^{\sigma}_{G/N} : \zeta(Q^F(G))^\times \rightarrow \zeta(Q^F(G/N))^\times,
\]
\[
\text{res}^{\sigma}_{H} : \zeta(Q^F(G))^\times \rightarrow \zeta(Q^F(H))^\times.
\]
The first map is easily seen to be induced by the canonical projection \(Q^F(G) \rightarrow Q^F(G/N)\). Moreover, by (an obvious generalisation of) [RW04, Lemma 9] we have commutative diagrams
\[
\begin{array}{c}
K_1(Q^F(G)) \xrightarrow{\text{nr}} \zeta(Q^F(G))^\times \\
\downarrow \text{quot}^{\sigma}_{G/N} \downarrow \\
K_1(Q^F(G/N)) \xrightarrow{\text{nr}} \zeta(Q^F(G/N))^\times
\end{array}
\]
\[
\begin{array}{c}
K_1(Q^F(G)) \xrightarrow{\text{nr}} \zeta(Q^F(G))^\times \\
\downarrow \text{res}^{\sigma}_{H} \\
K_1(Q^F(H)) \xrightarrow{\text{nr}} \zeta(Q^F(H))^\times.
\end{array}
\]

4. The equivariant Iwasawa main conjecture

4.1. Admissible one dimensional p-adic Lie extensions. Let \(p\) be an odd prime and let \(K\) be a totally real number field. We henceforth assume that \(L/K\) is an admissible one-dimensional \(p\)-adic Lie extension. In other words, \(L\) is a Galois extension of \(K\) such that (i) \(L\) is totally real, (ii) \(L\) contains the cyclotomic \(\mathbb{Z}_p\)-extension \(K_{\infty}\) of \(K\), and (iii) \([L : K_{\infty}]\) is finite. Let \(G = \text{Gal}(L/K)\), let \(H = \text{Gal}(L/K_{\infty})\) and let \(\Gamma_K = \text{Gal}(K_{\infty}/K)\). Let \(\gamma_K\) be a topological generator of \(\Gamma_K\). As in §3.3, we obtain a semidirect product \(G = H \times \Gamma\) where \(\Gamma \leq G\) and \(\Gamma \simeq \Gamma_K \simeq \mathbb{Z}_p\), and we choose an open subgroup \(\Gamma_0 \leq \Gamma\) that is central in \(G\).

4.2. An Iwasawa module and the \(\mu = 0\) hypothesis. Let \(S_\infty\) be the set of archimedean places of \(K\) and let \(S_p\) be the set of places of \(K\) above \(p\). Let \(S\) be a finite set of places of \(K\) containing \(S_p \cup S_\infty\). Let \(M_S^{ab}(p)\) be the maximal abelian pro-\(p\)-extension of \(L\) unramified outside \(S\) and let \(X_S = X_S(L/K) = \text{Gal}(M_S^{ab}(p)/L)\). As usual \(G\) acts on \(X_S\) by \(g \cdot x = \tilde{g}x\tilde{g}^{-1}\), where \(g \in G\), and \(\tilde{g}\) is any lift of \(g\) to \(\text{Gal}(M_S^{ab}(p)/K)\). This action extends to a left action of \(\Lambda(G)\) on \(X_S\). Since \(L\) is totally real, a result of Iwasawa [Iwa73] (also see [NSW08, Chapter XI, §3]) shows that \(X_S\) is finitely generated and torsion as a \(\Lambda(\Gamma_0)\)-module.

Definition 4.1. We say that \(L/K\) satisfies the \(\mu = 0\) hypothesis if \(X_S\) is finitely generated as a \(\mathbb{Z}_p\)-module.

The \(\mu = 0\) hypothesis is independent of the choice of \(S\) and is conjecturally always true. Moreover, it is known to hold when \(L/Q\) is abelian as follows from work of Ferrero and Washington [FW79]. For the relation to the classical Iwasawa \(\mu = 0\) conjecture see [JN18, Remark 4.3], for instance. In the sequel, we shall not assume the \(\mu = 0\) hypothesis for \(L/K\) except where explicitly stated.

4.3. The \(p\)-adic cyclotomic character and its projections. Let \(\chi_{\text{cyc}}\) be the \(p\)-adic cyclotomic character
\[
\chi_{\text{cyc}} : \text{Gal}(L(\zeta_p)/K) \rightarrow \mathbb{Z}_p^\times,
\]
defined by \(\sigma(\zeta) = \zeta^{\chi_{\text{cyc}}(\sigma)}\) for any \(\sigma \in \text{Gal}(L(\zeta_p)/K)\) and any \(p\)-power root of unity \(\zeta\). Let \(\omega\) and \(\kappa\) denote the composition of \(\chi_{\text{cyc}}\) with the projections onto the first and second factors of the canonical decomposition \(\mathbb{Z}_p^\times = \mu_{p-1} \times (1 + p\mathbb{Z}_p)\), respectively; thus \(\omega\) is the Teichmüller character. We note that \(\kappa\) factors through \(\Gamma_K\) (and thus also through \(G\)) and by abuse of notation we also use \(\kappa\) to denote the associated maps with these domains.
For $r \in \mathbb{N}_0$ divisible by $p - 1$ (or more generally divisible by the degree $[\mathcal{L}(\zeta_p) : \mathcal{L}]$), up to the natural inclusion map of codomains, we have $\chi_{cyc}^r = \kappa^r$.

4.4. A canonical complex. Let $S$ be a finite set of places of $K$ containing $S_p \cup S_\infty$. Let $\mathcal{O}_{\mathcal{L},S}$ denote the ring of integers $\mathcal{O}_\mathcal{L}$ in $\mathcal{L}$ localised at all primes above those in $S$. There is a canonical complex

$$C^*_S(\mathcal{L}/K) := R\text{Hom}_{\mathbb{Z}_p}(R\Gamma_\text{ét}(\text{Spec}(\mathcal{O}_{\mathcal{L},S}), \mathbb{Q}_p/\mathbb{Z}_p), \mathbb{Q}_p/\mathbb{Z}_p),$$

where $\mathbb{Q}_p/\mathbb{Z}_p$ denotes the constant sheaf of the abelian group $\mathbb{Q}_p/\mathbb{Z}_p$ on the étale site of Spec$(\mathcal{O}_{\mathcal{L},S})$. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is a direct limit of finite abelian groups of $p$-power order, we have an isomorphism with Galois cohomology (apply [Mil06, Chapter II, Proposition 2.9] and [Mil80, Chapter III, Lemma 1.16], for instance)

$$R\Gamma_\text{ét}(\text{Spec}(\mathcal{O}_{\mathcal{L},S}), \mathbb{Q}_p/\mathbb{Z}_p) \simeq R\Gamma(\text{Gal}(M_S(p)/\mathcal{L}), \mathbb{Q}_p/\mathbb{Z}_p),$$

where $M_S(p)$ is the maximal pro-$p$-extension of $\mathcal{L}$ unramified outside $S$. The cohomology groups of $C^*_S(\mathcal{L}/K)$ are

$$H^i(C^*_S(\mathcal{L}/K)) \cong \begin{cases} X_S & \text{if } i = -1 \\ \mathbb{Z}_p & \text{if } i = 0 \\ 0 & \text{if } i \neq 1,0. \end{cases}$$

(4.1)

Note that $C^*_S(\mathcal{L}/K)$ and the complex used by Ritter and Weiss (as constructed in [RW04]) become isomorphic in $\mathcal{D}(\Lambda(\mathcal{G}))$ by [Nic13, Theorem 2.4] (see also [Ven13] for more on this topic). Hence it makes no real difference which of these two complexes we use.

Let $S_{\text{ram}} = S_{\text{ram}}(\mathcal{L}/K)$ be the (finite) set of places of $K$ that ramify in $\mathcal{L}/K$. Note that since $\mathcal{L}$ contains the cyclotomic $\mathbb{Z}_p$-extension $\mathcal{L}_\infty$ we must have $S_p \subseteq S_{\text{ram}}$. The following result is well known, but we include a proof for the convenience of the reader.

Proposition 4.2. Suppose that $S$ contains $S_{\text{ram}} \cup S_\infty$.

(i) $C^*_S(\mathcal{L}/K)$ belongs to $\mathcal{D}_{\text{perf}}(\Lambda(\mathcal{G}))$.

(ii) Let $N$ be a finite normal subgroup of $\mathcal{G}$ and put $\mathcal{L}' := \mathcal{L}^N$. Then

$$\text{quot}^{\mathcal{G}}_{\mathcal{G}/N}([C^*_S(\mathcal{L}/K)]) = [C^*_S(\mathcal{L}'/K)].$$

(iii) Let $\mathcal{H}$ be an open subgroup of $\mathcal{G}$ and put $K' := \mathcal{L}^\mathcal{H}$. Then

$$\text{res}^{\mathcal{G}}_{\mathcal{G}/\mathcal{H}}([C^*_S(\mathcal{L}/K)]) = [C^*_S(\mathcal{L}'/K')],$$

where $S'$ is the set of places of $K'$ lying above those in $S$.

Proof. Let $G_{K,S} = \text{Gal}(K/S/K)$ where $K_S$ is the maximal algebraic extension of $K$ that is unramified outside the primes in $S$. Note that $\mathcal{G}$ is a quotient of $G_{K,S}$ since $S$ contains $S_{\text{ram}}$. Let $\Lambda(\mathcal{G})^\#(1)$ be the free $\Lambda(\mathcal{G})$-module of rank one upon which $\sigma \in G_{K,S}$ acts on the right via multiplication by the element $\chi_{cyc}(\sigma)\overline{\sigma}^{-1}$, where $\overline{\sigma}$ denotes the image of $\sigma$ in $\mathcal{G}$. Observe that by the middle row of [Lim12, Theorem on p. 2638], the isomorphism $\Lambda(\mathcal{G})^\#(1)^\vee(1) \cong (\Lambda(\mathcal{G})^\#)^\vee$ and a Shapiro lemma argument, we have

$$R\Gamma_c(\mathcal{O}_{K,S}, \Lambda(\mathcal{G})^\#(1))[3] \simeq R\text{Hom}_{\mathbb{Z}_p}(R\Gamma(G_{K,S}, (\Lambda(\mathcal{G})^\#)^\vee), \mathbb{Q}_p/\mathbb{Z}_p) \simeq C^*_S(\mathcal{L}/K)$$

in $\mathcal{D}(\Lambda(\mathcal{G}))$, where the left-hand side denotes the compact support cohomology complex with coefficients in $\Lambda(\mathcal{G})^\#(1)$. Thus $C^*_S(\mathcal{L}/K)$ is perfect by [FK06, Proposition 1.6.5]. Moreover, the cohomology groups of $C^*_S(\mathcal{L}/K)$ are torsion as $\Lambda(\Gamma_0)$-modules. Therefore (i) holds. Furthermore, (4.2) and loc. cit. together imply that there is an isomorphism

$$\Lambda(\mathcal{G}/N) \otimes_{\Lambda(\mathcal{G})}^L C^*_S(\mathcal{L}/K) \simeq C^*_S(\mathcal{L}'/K)$$
in \(D(\Lambda(\mathcal{G}/N))\), which gives part (ii). Part (iii) is clear. \(\square\)

Proposition 4.2 (i) implies that \(C^*_S(\mathcal{L}/K)\) defines a class \([C^*_S(\mathcal{L}/K)]\) in \(K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G}))\).

4.5. **Power series and \(p\)-adic Artin \(L\)-functions.** Recall that \(S\) is a finite set of places of \(K\) containing \(S_p \cup S_\infty\). Fix a character \(\chi \in \text{Irr}_{\mathbb{Q}_p}(\mathcal{G})\). Each topological generator \(\gamma_K\) of \(\Gamma_K\) permits the definition of a power series \(G_{\chi,S}(T) \in \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} \text{Quot}(\mathbb{Z}_p[T])\) by starting out from the Deligne-Ribet power series for one-dimensional characters of open subgroups of \(\mathcal{G}\) (see [DR80]; also see [Bar78, CN79]) and then extending to the general case by using Brauer induction (see [Gre83]). We put \(u := \kappa(\gamma_K)\). One then has an equality

\[
L_{p,S}(1 - s, \chi) = \frac{G_{\chi,S}(u^s - 1)}{H_{\chi}(u^s - 1)},
\]

where \(L_{p,S}(s, \chi)\) denotes the ‘\(S\)-truncated \(p\)-adic Artin \(L\)-function’ attached to \(\chi\) constructed by Greenberg [Gre83], and where, for irreducible \(\chi\), one has

\[
H_{\chi}(T) = \begin{cases} \chi(\gamma_K)(1 + T) - 1 & \text{if } H \subseteq \ker \chi \\ 1 & \text{otherwise.} \end{cases}
\]

Now [RW04, Proposition 11] implies that

\[
L_{K,S} : \chi \mapsto \frac{G_{\chi,S}(\gamma_K - 1)}{H_{\chi}(\gamma_K - 1)}
\]

is independent of the topological generator \(\gamma_K\) and lies in \(\text{Hom}^*_{\mathcal{G}_0}(R_p(\mathcal{G}), \mathbb{Q}^c(\Gamma_K)^\times)\). Diagram (3.8) implies that there is a unique element \(\Phi_S = \Phi_S(\mathcal{L}/K) \in \zeta(\mathcal{Q}(\mathcal{G}))^\times\) such that

\[
j_{\chi}(\Phi_S) = L_{K,S}(\chi)
\]

for every \(\chi \in \text{Irr}_{\mathbb{Q}_p}(\mathcal{G})\). The following result is a special case of [RW04, Proposition 12].

**Proposition 4.3.** (i) Let \(N\) be a finite normal subgroup of \(\mathcal{G}\) and put \(\mathcal{L}' := \mathcal{L}^N\). Then

\[
\text{quot}_{\mathcal{G}/N}^\mathcal{G}(\Phi_S(\mathcal{L}/K)) = \Phi_S(\mathcal{L}'/K).
\]

(ii) Let \(\mathcal{H}\) be an open subgroup of \(\mathcal{G}\) and put \(K' := \mathcal{L}/\mathcal{H}\). Then

\[
\text{res}_{\mathcal{H}}^\mathcal{G}(\Phi_S(\mathcal{L}/K)) = \Phi_{S'}(\mathcal{L}/K'),
\]

where \(S'\) is the set of places of \(K'\) lying above those in \(S\).

4.6. **Statement and known cases of the EIMC.** Recall that \(p\) is an odd prime and \(\mathcal{L}/K\) is an admissible one-dimensional \(p\)-adic Lie extension. Let \(S\) be a finite set of places of \(K\) containing \(S_{\text{ram}} \cup S_\infty\).

**Conjecture 4.4 (EIMC).** There exists \(\zeta_S \in K_1(\mathcal{Q}(\mathcal{G}))\) such that \(\partial(\zeta_S) = -[C^*_S(\mathcal{L}/K)]\) and \(\text{nr}(\zeta_S) = \Phi_S\).

It can be shown that the truth of Conjecture 4.4 is independent of the choice of \(S\), provided that \(S\) is finite and contains \(S_{\text{ram}} \cup S_\infty\). Crucially, this version of the EIMC does not require the \(\mu = 0\) hypothesis for its formulation. The following theorem has been shown independently by Ritter and Weiss [RW11] and by Kakte [Kak13].

**Theorem 4.5.** If \(\mathcal{L}/K\) satisfies the \(\mu = 0\) hypothesis then the EIMC holds for \(\mathcal{L}/K\).

By considering the cases in which the \(\mu = 0\) hypothesis is known, we obtain the following corollary (see [JN18, Corollary 4.6] for further details).
Corollary 4.6. Let $\mathcal{P}$ be a Sylow $p$-subgroup of $\mathcal{G}$. If $\mathcal{L}^\mathcal{P}/\mathbb{Q}$ is abelian then the EIMC holds for $\mathcal{L}/K$.

We shall also consider the EIMC with its uniqueness statement.

Conjecture 4.7 (EIMC with uniqueness). There exists a unique $\zeta_S \in K_1(\mathcal{Q}(\mathcal{G}))$ such that $\operatorname{nr}(\zeta_S) = \Phi_S$. Moreover, $\partial(\zeta_S) = -[C_S^*](\mathcal{L}/K)]$.

Remark 4.8. If $SK_1(\mathcal{Q}(\mathcal{G})) := \ker(\operatorname{nr} : K_1(\mathcal{Q}(\mathcal{G})) \to \zeta(\mathcal{Q}(\mathcal{G}))^\times)$ vanishes then it is clear that the uniqueness statement of the EIMC follows from its existence statement.

In [JN18], the present authors proved the EIMC unconditionally for an infinite class of one-dimensional admissible extensions for which the $\mu = 0$ hypothesis is not known to be true. However, such extensions must satisfy certain rather restrictive hypotheses, which, in particular, imply that $C_S^* (\mathcal{L}/K)$ may be replaced by any perfect complex with the cohomology specified in (4.1). We now recall the special case of these results given by [JN18, Theorem 4.12], whose proof relies crucially on a result of Ritter and Weiss [RW04, Theorem 16].

Theorem 4.9. If $p \nmid |H|$ then the EIMC with uniqueness holds for $\mathcal{L}/K$.

4.7. The EIMC for $\mathcal{L}/K$ implies the EIMC for all admissible subextensions.

The following result is well known, but we include a proof of the convenience of the reader.

Lemma 4.10. Let $p$ be an odd prime and let $\mathcal{L}/K$ be an admissible one-dimensional $p$-adic Lie extension of a totally real number field $K$. If the EIMC holds for $\mathcal{L}/K$ then the EIMC holds for all admissible sub-extensions of $\mathcal{L}/K$.

Proof. It suffices to show the result for admissible sub-extensions of the form $\mathcal{L}'/K$ and of the form $\mathcal{L}/K'$. We shall only prove the former case as the proof of the latter case is entirely analogous. Let $S$ be a finite set of places of $K$ containing $S_\infty \cup S_p \cup S_{\text{ram}}(\mathcal{L}/K)$. Since the EIMC holds for $\mathcal{L}/K$, there exists $\zeta_S \in K_1(\mathcal{Q}(\mathcal{G}))$ such that $\partial(\zeta_S) = -[C_S^* (\mathcal{L}/K)]$ and $\operatorname{nr}(\zeta_S) = \Phi_S(\mathcal{L}/K)$. Let $N = \text{Gal}(\mathcal{L}/\mathcal{L}')$. Specialising (3.5) and combining with the diagram on the left of (3.9) we obtain a commutative diagram

$$
\begin{array}{cccccc}
\zeta(\mathcal{Q}(\mathcal{G}))^\times & \xrightarrow{\text{nr}} & K_1(\mathcal{Q}(\mathcal{G})) & \xrightarrow{\partial} & K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G})) & \to 0 \\
\text{quot}_{\mathcal{G}/N}^\mathcal{S} & \downarrow & \xrightarrow{\text{quot}_{\mathcal{G}/N}^\mathcal{S}} & K_1(\mathcal{Q}(\mathcal{G}/N)) & \xrightarrow{\partial} & K_0(\Lambda(\mathcal{G}/N), \mathcal{Q}(\mathcal{G}/N)) & \to 0.
\end{array}
$$

Moreover, Propositions 4.2 and 4.3 give

$$
\text{quot}_{\mathcal{G}/N}^\mathcal{S}([C_S^* (\mathcal{L}/K)]) = [C_S^* (\mathcal{L}'/K)] \quad \text{and} \quad \text{quot}_{\mathcal{G}/N}^\mathcal{S} (\Phi_S(\mathcal{L}/K)) = \Phi_S(\mathcal{L}'/K).
$$

Therefore quot $\text{quot}_{\mathcal{G}/N}^\mathcal{S} (\zeta_S)$ has the desired properties and so the EIMC holds for $\mathcal{L}'/K$. \quad \Box

5. Some commutative algebra

5.1. Fitting ideals. If $M$ is a finitely presented module over a commutative ring $R$, we denote the (initial) Fitting ideal of $M$ over $R$ by $\operatorname{Fitt}_R(M)$. For basic properties of Fitting ideals including the following two well-known lemmas we refer the reader to Northcott’s excellent book [Nor76].

Lemma 5.1. Let $R$ be a commutative ring and let $M_1$ and $M_2$ be finitely presented $R$-modules. Then $\operatorname{Fitt}_R(M_1 \oplus M_2) = \operatorname{Fitt}_R(M_1) \operatorname{Fitt}_R(M_2)$. 
Lemma 5.2. Let $R \to S$ be a homomorphism of commutative rings and let $M$ be a finitely presented $R$-module. Then $S \otimes_R M$ is a finitely presented $S$-module and we have
\[
\text{Fitt}_S(S \otimes_R M) = S \otimes_R \text{Fitt}_R(M).
\]

In particular, Lemma 5.2 implies that Fitting ideals behave well under localisation.

Lemma 5.3. Let $p$ be a prime and let $G$ be a finite abelian group. Let $e \in \mathbb{Z}_p[G]$ be an idempotent such that $e = e^\#$. Let
\[
0 \longrightarrow M \longrightarrow C \longrightarrow C' \longrightarrow M' \longrightarrow 0
\]
be an exact sequence of finite $e\mathbb{Z}_p[G]$-modules and assume that $C$ and $C'$ are of finite projective dimension. Then one has an equality
\[
\text{Fitt}_{e\mathbb{Z}_p[G]}(M')^\# \cdot \text{Fitt}_{e\mathbb{Z}_p[G]}(C') = \text{Fitt}_{e\mathbb{Z}_p[G]}(C) \cdot \text{Fitt}_{e\mathbb{Z}_p[G]}(M').
\]

Proof. This is an obvious generalization of [CG98, Proposition 6]. See also [Nic10, Proposition 5.3].

5.2. A lemma on integral extensions and principal ideals. The following lemma expands on an argument given on [Gre00, p. 526].

Lemma 5.4. Let $S$ be an integral extension of a commutative ring $R$. If $a, b \in R$ such that $b$ is a nonzerodivisor, $Ra \subseteq Rb$ and $Sa = Sb$ then in fact $Ra = Rb$.

Proof. Since $Ra \subseteq Rb$ there exists $c \in R$ such that $a = bc$. Then $Sbc = Sa = Sb$ so there exists $s \in S$ such that $b = bcs$. As $b$ is nonzerodivisor we have $1 = cs$. Thus $c \in R \cap S^\times$ and so $c \in R^\times$ by [AM69, Chapter 5, Exercise 5(i)]. Therefore $Ra = Rbc = Rb$, as desired.

5.3. Cohen-Macaulay rings. A commutative Noetherian local ring $R$ is called Cohen-Macaulay if the depth of $R$ (the maximum length of a regular sequence in $R$) is equal to the Krull dimension of $R$. More generally, a commutative ring is called Cohen-Macaulay if it is Noetherian and all of its localisations at maximal ideals are Cohen-Macaulay.

Lemma 5.5. Let $R$ be a Cohen-Macaulay ring. Let $a$ and $b$ be nonzerodivisors of $R$. Then $a \in bR$ if and only if $aR_q \subseteq bR_q$ for every height one prime ideal $q$ of $R$.

Proof. One direction is trivial. For the other direction, we adapt the proof of [Fla04, Lemma 5.3]. We can and do assume that $bR$ is a proper ideal of $R$. Since $R$ is Cohen-Macaulay and $b$ is a nonzerodivisor, all prime divisors $p_1, \ldots, p_n$ of $bR$ have height one by [Mat89, Theorem 17.6]. By assumption $aR_{p_i} \subseteq bR_{p_i}$ for all $i = 1, \ldots, n$. Hence $a \in \phi_i^{-1}(bR_{p_i})$ where $\phi_i : R \to R_{p_i}$ is the canonical map. By the primary decomposition of the ideal $bR$ given by [Mat89, Theorem 6.8] we have
\[
a \in \phi_1^{-1}(bR_{p_1}) \cap \cdots \cap \phi_n^{-1}(bR_{p_n}) = bR.
\]

6. Commutative Iwasawa algebras

6.1. Structure of commutative Iwasawa algebras. Suppose that $G$ is an abelian one-dimensional compact $p$-adic Lie group. Then $G = H \times \Gamma$ where $H$ is a finite abelian group and $\Gamma \simeq \mathbb{Z}_p$. Let $R = \mathbb{Z}_p[\Gamma]$. Then $\Lambda(G) = R[H]$ is a commutative $R$-order in the separable $\text{Quot}(R)$-algebra $\mathbb{Q}(G)$. Let $\mathcal{M}(G)$ denote the unique maximal $R$-order in $\mathbb{Q}(G)$ and note that $\mathcal{M}(G)$ is the integral closure of $\Lambda(G)$ in $\mathbb{Q}(G)$.

We can write $H = H' \times H_p$ where $H_p$ is the Sylow $p$-subgroup of $H$ and $H'$ is a subgroup of order coprime to $p$. For an irreducible character $\chi$ of $H'$ let $\mathcal{O}_\chi = \mathbb{Z}_p[\chi]$ and note that
this is a finite unramified extension of $\mathbb{Z}_p$. Then we have a decomposition of $R$-orders $\Lambda(\mathfrak{g}) \cong \prod \mathcal{O}_\chi[\Gamma \times H_p]$, where the product runs over $\text{Irr}(H')/\sim$, the set of all irreducible characters of $H'$ modulo Galois conjugation over $\mathbb{Q}_p$. Each ring $\mathcal{O}_\chi[\Gamma \times H_p] \cong \mathcal{O}_\chi[H_p][T]$ is local and noetherian and its maximal ideal $m_\chi$ is equal to the radical of the ideal generated by $p$ and $T$. Moreover, $\mathcal{O}_\chi[H_p][T]$ is Cohen-Macaulay since $m_\chi$ is the unique prime ideal of height 2 and $p,T$ is a regular sequence.

Now let $e$ be any idempotent element of $\Lambda(\mathfrak{g})$ and define
\begin{equation}
\Lambda := e\Lambda(\mathfrak{g}), \quad \mathcal{M} := e\mathcal{M}(\mathfrak{g}), \quad \text{and} \quad Q := eQ(\mathfrak{g}).
\end{equation}
Then $\Lambda$ and $\mathcal{M}$ are both $R$-orders in $Q$ and $\mathcal{M}$ is maximal. Moreover,
\begin{equation}
\Lambda \cong \prod_{\chi \in \mathcal{I}} \mathcal{O}_\chi[\Gamma \times H_p] \cong \prod_{\chi \in \mathcal{I}} \mathcal{O}_\chi[H_p][T],
\end{equation}
where $\mathcal{I}$ is some subset of $\text{Irr}(H')/\sim$ and thus $\Lambda$ is itself Cohen-Macaulay.

6.2. **Height one prime ideals.** A prime ideal $q$ of $\Lambda$ is said to be singular if $p \in q$ and regular otherwise.

**Lemma 6.1.** Let $a, b \in \Lambda$ be nonzerodivisors. Suppose that $a\mathcal{M} = b\mathcal{M}$ and that $a\Lambda_q \subseteq b\Lambda_q$ for every singular height one prime ideal $q$ of $\Lambda$. Then $a\Lambda = b\Lambda$.

**Proof.** Since $\mathcal{M}$ is an integral extension of $\Lambda$, by Lemma 5.4 it suffices to show that $a\Lambda \subseteq b\Lambda$. Since $\Lambda$ is a noetherian Cohen-Macaulay ring, by Lemma 5.5 it suffices to show that $a\Lambda_q \subseteq b\Lambda_q$ for every height one prime ideal $q$ of $\Lambda$. If $q$ is singular, this holds by assumption. If $q$ is regular then $\mathcal{M}_q = \Lambda_q$ so $a\Lambda_q = b\Lambda_q$ by the assumption that $a\mathcal{M} = b\mathcal{M}$. \hfill $\Box$

For a finitely generated $R$-module $M$ we let $\mu(M)$ denote its $\mu$-invariant. We recall that $\mu(M) = 0$ if and only if the $\mathbb{Z}_p$-torsion submodule of $M$ is finite.

**Lemma 6.2.** Let $M$ be a finitely generated $\Lambda$-module that is $R$-torsion. Then $\mu(M) = 0$ if and only if $M_q = 0$ for every singular height one prime ideal $q$ of $\Lambda$.

**Proof.** This follows from [BG03, Lemma 6.3], the decomposition (6.2) and the additivity of $\mu$-invariants with respect to short exact sequences. (Also see [Fla04, Lemma 5.6].) \hfill $\Box$

6.3. **Fitting ideals of Iwasawa modules.** The following lemma is well known.

**Lemma 6.3.** Let $M$ be a finitely generated $\Lambda$-module that is of projective dimension at most one and that is also $R$-torsion. Then $M$ has a quadratic presentation of the form
\begin{equation}
0 \longrightarrow \Lambda^n \xrightarrow{h} \Lambda^n \longrightarrow M \longrightarrow 0
\end{equation}
for some $n \in \mathbb{N}$ and $\text{Fitt}_\Lambda(M)$ is a principal ideal generated by a nonzerodivisor.

**Proof.** Let $0 \rightarrow P \rightarrow \Lambda^n \rightarrow M \rightarrow 0$ be a projective resolution of $M$. By [NSW08, (5.2.20)] $P$ is a direct sum of modules of the form $(\mathcal{O}_\chi[\Gamma \times H_p])^{n_\chi}$ for some $n_\chi \geq 0$. Since $M$ is $R$-torsion $n_\chi = n$ if $\chi \in \mathcal{I}$ and $n_\chi = 0$ otherwise. Thus we can take $P = \Lambda^n$ and so we have a presentation of the form (6.3). Thus $\text{Fitt}_\Lambda(M)$ is principal by definition of Fitting ideal and any generator is a nonzerodivisor since $h$ is injective. \hfill $\Box$

We recall the following result of Greither and Kurihara [GK08, Theorem 2.1]. We caution that the notation here differs from that of loc. cit. (the roles of $R$ and $\Lambda$ are reversed). Let $\gamma$ be a topological generator of $\Gamma$. For $n \geq 1$ define $\omega_n = \gamma^{p^n} - 1 \in R$ and $\Lambda_n = \Lambda/\omega_n\Lambda$. Then $(\Lambda_n)_n$ is a projective system with limit $\Lambda$ and we make the natural
identification \( \Lambda \cong \varprojlim_n \Lambda_n \). We shall consider projective systems \( (A_n)_n \) of modules \( A_n \) over \( \Lambda_n \) such that the transition maps \( A_m \to A_n \) \((m \geq n)\) are \( \Lambda_m \)-linear in the obvious sense. The limit \( M := \varprojlim_n A_n \) will then be a \( \Lambda \)-module.

**Theorem 6.4** (Greither and Kurihara). Suppose that the limit \( M \) is a finitely generated \( \Lambda \)-module that is \( R \)-torsion and that there exists \( n_0 \geq 1 \) such that the transition map \( A_m \to A_n \) is surjective for all \( m \geq n \geq n_0 \). Then \( \text{Fitt}_\Lambda(M) = \varprojlim_n (\text{Fitt}_n(\Lambda_n)A_n) \).

**Proof.** In [GK08, Theorem 2.1], this is stated in the case \( \Lambda = \Lambda(G) \). It is clear that this implies the desired result for any choice of \( \Lambda \) as defined in (6.1). \( \Box \)

Even though \( \Lambda \) is commutative in the present setting, for clarity we do not assume this in the following definition. For each (left) \( \Lambda \)-module \( M \) we set \( E^1(M) := \text{Ext}^1_R(M, R) \), which has a canonical right \( \Lambda \)-module structure. Set \( \Lambda^\# := \{ \lambda^\# \mid \lambda \in \Lambda \} = e^\# \Lambda \). Then \( E^1(M) \) is a left \( \Lambda^\# \)-module, as \( \lambda^\# \in \Lambda^\# \) acts on \( f \in E^1(M) \) by \( \lambda^\# f = f\lambda \).

**Lemma 6.5.** Let \( M \) be a finitely generated \( \Lambda \)-module that is of projective dimension at most one and that is also \( R \)-torsion. Then \( E^1(M) \) is a finitely generated \( \Lambda^\# \)-module of projective dimension at most one and is \( R \)-torsion. Moreover, we have an equality

\[
\text{Fitt}_{\Lambda^\#}(E^1(M)) = \text{Fitt}_{\Lambda}(M)^\#.
\]

**Proof.** By Lemma 6.3 we may choose a quadratic presentation of \( M \) as in (6.3). We apply the functor \( \text{Hom}_R(-, R) \) to this sequence. Since \( M \) is \( R \)-torsion and \( \Lambda \) is a projective \( R \)-module, we have \( \text{Hom}_R(M, R) = E^1(\Lambda) = 0 \). We identify \( \text{Hom}_R(\Lambda, R) \) and \( \Lambda^\# \) so that we obtain an exact sequence

\[
0 \longrightarrow (\Lambda^\#)^n h^{T^\#} \longrightarrow (\Lambda^\#)^n \longrightarrow E^1(M) \longrightarrow 0,
\]

where the second map is obtained from \( h \) by applying the involution \( \# \) to its transpose. The result follows. \( \Box \)

### 6.4. Fitting ideals of complexes

Since \( \Lambda \) and \( Q \) are both commutative semilocal rings, [CR87, (45.12)] shows that the determinant map induces isomorphisms \( K_1(\Lambda) \cong \Lambda^\times \) and \( K_1(Q) \cong Q^\times \). Using this fact, specialising (3.4) to the case \( \mathcal{O} = \mathbb{Z}_p \), and multiplying by the idempotent \( e \) gives a short exact sequence

\[
0 \longrightarrow K_1(\Lambda) \longrightarrow K_1(Q) \overset{\partial}{\longrightarrow} K_0(\Lambda, Q) \longrightarrow 0.
\]

Now let \( C^* \in \mathcal{D}_{tor}^\text{perf}(\Lambda) \) and recall from §3.2 that \( C^* \) defines an element \( [C^*] \) in \( K_0(\Lambda, Q) \). Choose \( x \in K_1(Q) \) such that \( \partial(x) = [C^*] \) and define

\[
\text{Fitt}_{\Lambda}(C^*) := \det(x)\Lambda.
\]

Note that this is well defined by the exactness of (6.4). If \( C_i^* \in \mathcal{D}_{tor}^\text{perf}(\Lambda) \) for \( i = 1, 2, 3 \) such that \( [C_2^*] = [C_1^*] + [C_3^*] \) in \( K_0(\Lambda, Q) \) (this is the case in the situation of (3.2), for example) then it is straightforward to show that

\[
\text{Fitt}_{\Lambda}(C_2^*) = \text{Fitt}_{\Lambda}(C_1^*) \cdot \text{Fitt}_{\Lambda}(C_3^*).
\]

The following is a special case of [JN19, Lemma 2.7].

**Lemma 6.6.** Let \( A \) and \( B \) be finitely generated \( R \)-torsion \( \Lambda \)-modules of projective dimension at most 1. Let \( A \to B \) be a complex concentrated in degrees \(-1\) and \( 0 \). Then

\[
\text{Fitt}_{\Lambda}(A \to B) = \text{Fitt}_{\Lambda}^{-1}(A) \cdot \text{Fitt}_{\Lambda}(B).
\]
Proof. We consider $A$ and $B$ as complexes concentrated in degree 0. Then we have a short exact sequence of complexes

$$0 \rightarrow B \rightarrow (A \rightarrow B) \rightarrow A[1] \rightarrow 0,$$

where $A[1]$ is concentrated in degree $-1$. Hence by (3.2) we have

$$[A \rightarrow B] = [B] + [A[1]] = [B] - [A],$$

in $K_0(\Lambda, \mathbb{Q})$ and so the desired result now follows from (6.6). \qed

7. Summary of relevant results from [JN19, §8]

7.1. Setup and notation. Let $p$ be an odd prime and let $K$ be a totally real number field. Let $L/K$ be a finite Galois CM extension (not necessarily abelian).

Assumptions. We henceforth assume that $S, T$ are finite sets of places of $K$ and that

(i) $\zeta_p \in L$, (ii) $S \cap T = \emptyset \neq T$, (iii) $S_p \cup \mathcal{S}_{\text{ram}}(L/K) \cup S_\infty \subseteq S$, and (iv) $\mathbb{Z}_p \otimes_{\mathbb{Z}} E^T_L$ is torsionfree.

Let $L_\infty$ and $K_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extensions of $L$ and $K$, respectively. Let $\mathcal{G} := \text{Gal}(L_\infty/K)$, which we write as $\mathcal{G} = H \rtimes \Gamma$ where $\Gamma \simeq \mathbb{Z}_p$ and $H := \text{Gal}(L_\infty/K_\infty)$ naturally identifies with a normal subgroup of $G$. Let $\Gamma_0$ be an open subgroup of $\Gamma$ that is central in $\mathcal{G}$ and recall from (3.3) that $\Lambda(\mathcal{G}) := \mathbb{Z}_p[\mathcal{G}]$ is a free $R := \mathbb{Z}_p[\Gamma_0]$-order in $\mathcal{Q}(\mathcal{G})$. Let $j \in \mathcal{G}$ denote complex conjugation (this an abuse of notation because its image in the quotient group $G := \text{Gal}(L/K)$ is also denoted by $j$) and let $\mathcal{G}^+ := \mathcal{G}/(j) = \mathcal{G}(L_\infty^+/K)$. Then $j \in H$ and so again $\Lambda(\mathcal{G}^+)$ is a free $R$-order in $\mathcal{Q}(\mathcal{G}^+)$. Moreover, $\Lambda(\mathcal{G})_+ := \Lambda(\mathcal{G})/(1+j)$ is also a free $R$-order. For any $\Lambda(\mathcal{G})$-module $M$ we write $M^+$ and $M^-$ for the submodules of $M$ upon which $j$ acts as $1$ and $-1$, respectively, and consider these as modules over $\Lambda(\mathcal{G})^+$ and $\Lambda(\mathcal{G})_-$, respectively. We note that $M$ is $R$-torsion if and only if both $M^+$ and $M^-$ are $R$-torsion.

Let $\chi_{\text{cyc}} : \mathcal{G} \rightarrow \mathbb{Z}_p^\times$ denote the $p$-adic cyclotomic character. Let $\mu_{p^n} = \mu_{p^n}(L_\infty)$ denote the group of $p^n$th roots of unity in $L_\infty^\times$ and let $\mu_{p,\infty}$ be the nested union (or direct limit) of these groups. Let $\mathbb{Z}_p(1) := \varprojlim_{n} \mu_{p^n}$ be endowed with the action of $\mathcal{G}$ given by $\chi_{\text{cyc}}$. For any $r \geq 0$ define $\mathbb{Z}_p(r) := \mathbb{Z}_p(1)^{\otimes r}$ and $\mathbb{Z}_p(-r) := \text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(r), \mathbb{Z}_p)$ endowed with the naturally associated actions. For any $\Lambda(\mathcal{G})$-module $M$, we define the $r$th Tate twist to be $M(r) := \mathbb{Z}_p(r) \otimes_{\mathbb{Z}_p} M$ with the natural $\mathcal{G}$-action; hence $M(r)$ is simply $M$ with the modified $\mathcal{G}$-action $g \cdot m = \chi_{\text{cyc}}(g)^r g(m)$ for $g \in \mathcal{G}$ and $m \in M$. In particular, we have $\mathbb{Q}_p/\mathbb{Z}_p(1) \simeq \mu_{p,\infty}$ and $\Lambda(\mathcal{G}^+)(-1) \simeq \Lambda(\mathcal{G})_-$. We note that the property of being $R$-torsion is preserved under taking Tate twists.

For every place $v$ of $K$ we denote the decomposition subgroup of $\mathcal{G}$ at a chosen prime $w_\infty$ above $v$ by $\mathcal{G}_{w_\infty}$ (everything will only depend on $v$ and not on $w_\infty$ in the following). We note that the index $[\mathcal{G} : \mathcal{G}_{w_\infty}]$ is finite when $v$ is a finite place of $K$. Let $\phi_{w_\infty}$ denote the Frobenius automorphism at $w_\infty$.

7.2. Statement of results. We now state the relevant results from [JN19, §8]. Let

$$I_T = \left( \bigoplus_{v \in T} \text{ind}_{\mathcal{G}_{w_\infty}}^{\mathcal{G}} \mathbb{Z}_p(-1) \right)^{-}.
$$

Recall that $X^+_S = X_S(L_\infty^+/K)$ and $C^+_S(L_\infty^+/K)$ were defined in §4.2 and §4.4, respectively. We let $A^T_{L_\infty}$ denote the direct limit $\lim_{\rightarrow n} A^T_{K_n}$. 

(7.1)
Proposition 7.1. There exists a $\Lambda(\mathcal{G})_-$-module $Y_S^T(-1)$ and a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X_S^+(-1) & \longrightarrow & Y_S^T(-1) & \longrightarrow & I_T & \longrightarrow & Z_p(-1) & \longrightarrow & 0 \\
\alpha & & \beta & & & & & & & \\
0 & \longrightarrow & X_S^+(1) & \longrightarrow & \text{Hom}(A_{L_\infty}^T, \mathbb{Q}_p/\mathbb{Z}_p) & \longrightarrow & I_T & \longrightarrow & Z_p(-1) & \longrightarrow & 0,
\end{array}
\]

with exact rows and columns where the middle two terms of the upper and lower rows (concentrated in degrees $-1$ and $0$) represent $C_S^\bullet(L_\infty^+/K)(-1)$ and $C_{S_p}^\bullet(L_\infty^+/K)(-1)$, respectively. Moreover, both $I_T$ and $Y_S^T(-1)$ are of projective dimension at most $1$ over $\Lambda(\mathcal{G})_-$ and are $R$-torsion.

Proof. See [JN19, Lemma 8.4, Proposition 8.5 and Lemma 8.6].

Remark 7.2. The proof of the existence of the map $\beta$ in diagram (7.2) relies on the fact that the middle two terms of each row represent the correct complex. Thus the classes of these complexes in $D(\Lambda(\mathcal{G}))$ play a crucial role.

8. THE PROOF THEOREM 1.1

We keep the notation and assumptions of §7, but now specialise to the abelian case. Thus, in particular, $L/K$ is a finite abelian CM extension such that $\zeta_p \in L$. We make the further assumption that $L \cap K_\infty = K$. Then we have $\mathcal{G} = H \times \Gamma$ where $\Gamma = \text{Gal}(L_\infty/L)$. Moreover, we take $\Gamma_0 = \Gamma$ and $R = \mathbb{Z}_p[\Gamma]$. For any $r \in \mathbb{Z}$ we let $t_{r,\text{cyc}}$ be the $\mathbb{Q}_p$-algebra automorphism of $\mathcal{Q}(\mathcal{G})$ induced by $g \mapsto \chi_{r,\text{cyc}}(g)g$ for $g \in \mathcal{G}$. This restricts to an $\mathbb{Z}_p$-algebra automorphism of $\Lambda(\mathcal{G})$ and for $r = 1$ induces an isomorphism $\Lambda(\mathcal{G}_+)(-1) \cong \Lambda(\mathcal{G})_-$. We define

$$\Psi_{S,T} = \Psi_{S,T}(L_\infty/K) := \theta_{1,\text{cyc}}(\Phi_S) \cdot \prod_{v \in T} \xi_v,$$

where $\xi_v := 1 - \chi_{r,\text{cyc}}(\phi_{w_v})\phi_{w_v}$.

Lemma 8.1. We have $\Psi_{S,T} = \lim_{n \to \infty} \theta_n^{\#}(L_n/K)^\#$.

Proof. This follows from [GP15, Lemma 5.14 (2)] with $m = 1$ after applying the involution $\#$. 

Lemma 8.2. We have $\text{Fitt}_{\Lambda(\mathcal{G})_-}(Y_S^T(-1)) = \left(\prod_{v \in T} \xi_v\right) \text{Fitt}_{\Lambda(\mathcal{G})_-}(C_S^\bullet(L_\infty^+/K)(-1))^{-1}$.

Proof. Applying Lemma 6.6 to the assertion of Proposition 7.1 gives

\[
\text{Fitt}_{\Lambda(\mathcal{G})_-}(Y_S^T(-1)) = \text{Fitt}_{\Lambda(\mathcal{G})_-}(C_S^\bullet(L_\infty^+/K)(-1))^{-1} \cdot \text{Fitt}_{\Lambda(\mathcal{G})_-}(I_T).
\]

For each place $w_\infty$ of $L_\infty$ we have an exact sequence of $\Lambda(\mathcal{G}_{w_\infty})$-modules

$$0 \longrightarrow \Lambda(\mathcal{G}_{w_\infty}) \longrightarrow \Lambda(\mathcal{G}_{w_\infty}) \longrightarrow Z_{p,w_\infty}(-1) \longrightarrow 0,$$

where the injection is multiplication by $1 - \chi_{r,\text{cyc}}(\phi_{w_\infty})\phi_{w_\infty}$. Thus

\[
\text{Fitt}_{\Lambda(\mathcal{G})_-}(I_T) = \prod_{v \in T} \text{Fitt}_{\Lambda(\mathcal{G})_-} \left( \text{ind}_{\mathcal{G}_{w_\infty}}^\mathcal{G} Z_{p,-}(1) \right)^{-1} = \left(\prod_{v \in T} \xi_v\right) \Lambda(\mathcal{G})_-,
\]

where the first equality follows from the Lemma 5.1 and the definition of $I_T$. The desired result now follows by combining (8.1) and (8.2). 

The following is similar to [JN19, Proposition 8.7].
Proposition 8.3. The EIMC holds for $L^+_{\infty}/K$ if and only if $\Psi_{S,T}$ is a generator of $\text{Fitt}_{\mathcal{A}(G)_-}(Y^T_{S}(-1))$.

Proof. Since $G^+$ is abelian the reduced norm map $n_r : K_1(\mathcal{O}(G^+)) \to \mathcal{O}(G^+)^\times$ is equal to the usual determinant map and is an isomorphism by [CR87, (45.12)]. Thus it is straightforward to see that the EIMC for $L$ to the usual determinant map $\det$ and is an isomorphism by [CR87, (45.12)]. Thus it is the same as asserting that $\Phi_{S}^{-1}$ generates the Fitting ideal of $C^*_S(L^+_{\infty}/K) \in \mathcal{D}_{\text{tor}}(\Lambda(G^+))$. Since $t^1_{\text{cyc}}$ induces an isomorphism $\Lambda(G^+)(-1) \cong \Lambda(G)_-$, this in turn is equivalent to the assertion that $t^1_{\text{cyc}}(\Phi_{S})^{-1}$ generates the Fitting ideal of $C^*_S(L^+_{\infty}/K)(-1) \in \mathcal{D}_{\text{tor}}(\Lambda(G)_-)$. Therefore the desired result now follows from Lemma 8.2 and the definition of $\Psi_{S,T}$. \hfill $\square$

The following result is ultimately a reformulation of the classical Iwasawa main conjecture proven Wiles [Wil90].

Proposition 8.4. $\Psi_{S,T}$ is a generator of $\text{Fitt}_{\mathcal{M}(G)_-}(\mathcal{M}(G)_- \otimes_{\Lambda(G)_-} Y^T_{S}(-1))$.

Proof. By [JN18, Corollary 4.10] with $e = 1$ there exists an element $y_S \in K_1(\mathcal{O}(G^+))$ such that $n_r(y_S) = \Phi_S$ and $y_S$ maps to $[\mathcal{M}(G^+) \otimes_{\Lambda(G^+)} C^*_S(L^+_{\infty}/K)]$ under the map $K_1(\mathcal{O}(G^+)) \to K_0(\mathcal{M}(G^+), \mathcal{O}(G^+))$. But $n_r$ is an isomorphism since $G^+$ is abelian and so we must have $y_S = \zeta_S := n_r^{-1}(\Phi_S)$. By Lemma 5.2 we may extend scalars from $\Lambda(G)_-$ to $\mathcal{M}(G)_-$ in the statement of Lemma 8.2. Similarly, by extending scalars in the proof of Proposition 8.3 we obtain the desired result. \hfill $\square$

Theorem 8.5. The EIMC holds for $L^+_{\infty}/K$.

Proof. We abbreviate $\Lambda(G)_-$ to $\Lambda$ and $\mathcal{M}(G)_-$ to $\mathcal{M}$. By Proposition 7.1 and Lemma 6.3, $\text{Fitt}_\Lambda(Y^T_{S}(-1))$ is a principal ideal generated by a nonzerodivisor. Moreover, by Proposition 8.4 and [Eis95, Corollary 20.5] we have $\Psi_{S,T,\mathcal{M}} = \text{Fitt}_\Lambda(Y^T_{S}(-1))\mathcal{M}$. By Lemma 6.1 and Proposition 8.3 it suffices to show that

$$\Psi_{S,T,\Lambda_q} \subseteq \text{Fitt}_\Lambda(Y^T_{S}(-1))q = \text{Fitt}_\Lambda(Y^T_{S}(-1))q$$

for every singular height one prime ideal $q$ of $\Lambda$.

Now consider the commutative diagram (7.2). By [NSW08, (11.3.6)] we have $\mu(X^+_S) = \mu(X^+_S) = \mu(X^+_S) = \mu(X^+_S)$ and hence $\mu(X^+_{S}(-1)) = \mu(X^+_{S}(-1))$. Thus the additivity of $\mu$-invariants with respect to short exact sequences and the application of the snake lemma to (7.2) shows that $\mu(\ker(\alpha)) = \mu(\ker(\beta)) = 0$. Hence Lemma 6.2 gives

$$Y^T_{S}(-1)q \cong \text{Hom}(A^T_{L_{\infty}}; \mathbb{Q}_p/\mathbb{Z}_p)_q$$

for every singular height one prime ideal $q$ of $\Lambda$. Therefore it suffices to show that

$$\Psi_{S,T} \in \text{Fitt}_\Lambda(\text{Hom}(A^T_{L_{\infty}}; \mathbb{Q}_p/\mathbb{Z}_p)).$$

For $n \geq 0$ let $L_n$ denote the $n$th layer of $L_{\infty}$ and let $G_n = \text{Gal}(L_n/K)$. As the transition maps in the direct limit $A^T_{L_{\infty}} = \lim_{\leftarrow n} A^T_{L_n}$ are injective by [GP15, Lemma 2.9], the transition maps in the projective limit $\text{Hom}(A^T_{L_{\infty}}; \mathbb{Q}_p/\mathbb{Z}_p) = \lim_{\to n} (A^T_{L_n})^\vee$ are surjective. Thus by Theorem 6.4 we have

$$\text{Fitt}_\Lambda(\text{Hom}(A^T_{L_{\infty}}; \mathbb{Q}_p/\mathbb{Z}_p)) = \lim_{\to n} \text{Fitt}_{\mathbb{Z}_p[G_n]}((A^T_{L_n})^\vee).$$

By Theorem 2.3 we have $\theta^T_{S}(L_n/K)^\# \in \text{Fitt}_{\mathbb{Z}_p[G_n]}((A^T_{L_n})^\vee)$ for every $n \geq 0$. Therefore (8.3) now follows from Lemma 8.1 and (8.4). \hfill $\square$
Lemma 8.6. Let $p$ be an odd prime and let $K$ be a totally real number field. Let $\mathcal{L}/K$ be an abelian admissible one-dimensional $p$-adic Lie extension of $K$. There exists a finite abelian CM extension $L/K$ such that (i) $\zeta_p \in L$, (ii) $L \cap K_\infty = K$ and (iii) $L_\infty^+$ contains $L$.

Proof. Let $G = \text{Gal}(\mathcal{L}/K)$, let $H = \text{Gal}(\mathcal{L}/K_\infty)$ and let $\Gamma_K = \text{Gal}(K_\infty/K)$. As in §4.1 we obtain a semidirect product $G = H \rtimes \Gamma$ where $\Gamma \leq G$ and $\Gamma \simeq \Gamma_K \simeq \mathbb{Z}_p$. Since $G$ is abelian the product is in fact direct. Let $F$ be the subfield of $L$ fixed by $\Gamma$. Then $F_\infty = L$ and $F \cap K_\infty = K$. Now let $L = F(\zeta_p)$. Then $L/K$ is a finite abelian CM extension and $L$ satisfies properties (i), (ii) and (iii). We note that the choice of $\Gamma$ and hence of $L$ is non-canonical. □

Proof of Theorem 1.1. Let $L/K$ be as in Lemma 8.6. Theorem 8.5 says that the EIMC holds for $L_\infty^+/K$ and so the EIMC also holds for $L/K$ by Lemma 4.10. Let $G = \text{Gal}(\mathcal{L}/K)$. Since $G$ is abelian, the reduced norm map $\text{nr} : K_1(\mathbb{Q}(G)) \to \mathbb{Q}(G)^\times$ is an isomorphism by [CR87, (45.12)] and so $SK_1(\mathbb{Q}(G)) = 0$. Hence we also have uniqueness (see Remark 4.8). □

9. Iwasawa algebras and commutator subgroups

The following theorem is a restatement of a special case of [JN13, Proposition 4.5]. We include the proof here for the convenience of the reader and take the opportunity to correct some minor oversights in the proof of loc. cit.

Theorem 9.1. Let $p$ be a prime, let $G$ be a one-dimensional compact $p$-adic Lie group and let $F/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$. The commutator subgroup $G'$ of $G$ is finite and $\Lambda^{\mathcal{O}}(G)$ is a direct product of matrix rings over (complete local) commutative rings if and only if $p \nmid |G'|$.

Proof. We adopt the setup and notation of §3.3. We identify $R$ with $\mathcal{O}_J$ and abbreviate $\Lambda^{\mathcal{O}}(G)$ to $\Lambda$. Let $p$ and $P$ denote the maximal ideals of $\mathcal{O}$ and $R$, respectively. Then $P$ is generated by $p$ and $T$. Let $k = R/P = \mathcal{O}/p$ be the residue field, which is finite and of characteristic $p$. Let $C_{p^n}$ denote the cyclic group of order $p^n$. Since $\gamma^{p^n} = 1 + T \equiv 1 \mod p$, we have

$$\Lambda := \Lambda/\mathfrak{P} \Lambda = \bigoplus_{i=0}^{p^n-1} k[H]\gamma^i = k[H \rtimes C_{p^n}] \cong k \otimes_R \Lambda.$$  

(9.1)

Since $G/H \simeq \Gamma$ is abelian, $G'$ is actually a subgroup of $H$ and thus is finite. Moreover, $G'$ identifies with the commutator subgroup of $H \rtimes C_{p^n}$.

We refer the reader to [AG60] for background on separability and recall that a ring is said to be an Azumaya algebra if it is separable over its centre. We shall show that the following assertions are equivalent.

(i) $\Lambda$ is a direct product of matrix rings over (complete local) commutative rings;  
(ii) $\Lambda$ is a direct product of matrix rings over commutative rings;  
(iii) $\Lambda$ is an Azumaya algebra;  
(iv) $\Lambda$ is an Azumaya algebra;  
(v) $p \nmid |G'|$.

As any matrix ring over a commutative ring is an Azumaya algebra, (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv). In fact, as remarked after [DJ83, Corollary, p. 390] we have (ii) $\iff$ (iv). By [DJ83, Corollary p. 389] we have (iv) $\iff$ (v).
We now show (iii) ⇔ (iv). By [Lam01, Example 23.3] \( \zeta(\Lambda) \) is semiperfect and thus a product of local rings by [Lam01, Theorem 23.11], say \( \zeta(\Lambda) = \bigoplus_{i=1}^{r} R_i \), where each \( R_i \) contains \( R \). By [CR81, Proposition 6.5 (ii)] each \( R_i \) is in fact a complete local ring. Let \( \mathfrak{p}_i \) be the maximal ideal of \( R_i \) and \( k_i := R_i/\mathfrak{p}_i \) be the residue field. Note that we have

\[
(9.2) \quad \zeta(\overline{\Lambda}) = \zeta(\Lambda) \otimes_R k = \bigoplus_{i=1}^{r} R_i \otimes_R k = \bigoplus_{i=1}^{r} R_i/\mathfrak{p}_i.
\]

In order to justify the first equality, we observe that it easily follows from the decomposition (3.3) that the centre \( \zeta(\Lambda) \) is a free \( R \)-module of rank \( c(\mathcal{G}/\Gamma_0) \), where \( c(A) \) denotes the number of conjugacy classes of a group \( A \); a basis is given by the class sums. Similarly, it follows from (9.1) that \( \zeta(\overline{\Lambda}) \) is a \( k \)-vector space of dimension \( c(H \rtimes C_{p^r}) = c(\mathcal{G}/\Gamma_0) \).

Hence the obvious inclusion \( \zeta(\Lambda) \otimes_R k \subseteq \zeta(\overline{\Lambda}) \) must be an equality.

Moreover, we also have

\[
(9.3) \quad \Lambda \otimes_{\zeta(\Lambda)} k_i = \Lambda \otimes_{R_i} k_i \cong (\Lambda \otimes_R k) \otimes_{(R_i \otimes_R k)} (k_i \otimes_R k) \cong \overline{\Lambda} \otimes_{\zeta(\overline{\Lambda})} k_i.
\]

By [AG60, Theorem 4.7] \( \Lambda \) is Azumaya if and only if \( \Lambda \otimes_{\zeta(\Lambda)} k_i \) is separable over \( k_i \) for each \( i \). Similarly, by (9.3) and loc. cit. \( \overline{\Lambda} \) is Azumaya if and only if \( \overline{\Lambda} \otimes_{\zeta(\overline{\Lambda})} k_i \) is separable over \( k_i \) for each \( i \). Therefore the claim now follows from (9.2).

In summary, we have shown that (ii) ⇔ (iii) ⇔ (iv) ⇔ (v) and (i) ⇒ (iii). Thus it remains to show (iii) ⇒ (i). Suppose (iii) holds. Since \( \mathfrak{p}_i R_i \subseteq \mathfrak{p}_i \), the natural projection \( R_i \to k_i \) factors through \( R_i \to R_i/\mathfrak{p}_i R_i = R_i \otimes_R k_i \). Hence we have the corresponding homomorphisms of Brauer groups

\[
\text{Br}(R_i) \to \text{Br}(R_i/\mathfrak{p}_i R_i) \to \text{Br}(k_i).
\]

Now \( \text{Br}(R_i) \to \text{Br}(k_i) \) is injective by [AG60, Corollary 6.2] and hence \( \text{Br}(R_i) \to \text{Br}(R_i/\mathfrak{p}_i R_i) \) must also be injective. This yields an embedding

\[
\text{Br}(\zeta(\Lambda)) = \bigoplus_{i=1}^{r} \text{Br}(R_i) \hookrightarrow \bigoplus_{i=1}^{r} \text{Br}(R_i \otimes_R k) = \text{Br}(\zeta(\overline{\Lambda})).
\]

Since \( \Lambda \) is Azumaya, it defines a class \( [\Lambda] \in \text{Br}(\zeta(\Lambda)) \) which is mapped to \( [\overline{\Lambda}] \) via this embedding. In particular, (iv) holds and we have already seen that this implies (ii). Hence \( [\overline{\Lambda}] \) is trivial and thus so is \( [\Lambda] \). Let \( \Lambda_i \) be the component of \( \Lambda \) corresponding to \( R_i \). Then \( [\Lambda_i] \in \text{Br}(R_i) \) is trivial and so by [AG60, Proposition 5.3] \( \Lambda_i \) is isomorphic to an \( R_i \)-algebra of the form \( \text{Hom}_{R_i}(P_i, P_i) \) where \( P_i \) is a finitely generated projective faithful \( R_i \)-module. Since \( R_i \) is a local ring, \( P_i \) must be free and so \( \Lambda_i \) must be isomorphic to a matrix ring over its centre \( R_i \). Thus (i) holds. \( \square \)

**Corollary 9.2.** Let \( p \) be a prime and let \( \mathcal{G} \) be a one-dimensional compact \( p \)-adic Lie group such that \( p \nmid |\mathcal{G}| \). Let \( F/\mathbb{Q}_p \) be a finite extension with ring of integers \( \mathcal{O} \). Then \( \mathcal{Q}^F(\mathcal{G}) \) is a direct product of matrix rings over fields and there is a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & K_1(\mathcal{Q}^\mathcal{O}(\mathcal{G})) & \longrightarrow & K_1(\mathcal{Q}^F(\mathcal{G})) & \longrightarrow & K_0(\mathcal{Q}^\mathcal{O}(\mathcal{G}), \mathcal{Q}^F(\mathcal{G})) & \longrightarrow & 0 \\
& & \downarrow_{\cong} & & \downarrow_{\cong} & & \downarrow \cong & & \\
0 & \longrightarrow & \zeta(\mathcal{Q}^\mathcal{O}(\mathcal{G})) & \longrightarrow & \zeta(\mathcal{Q}^F(\mathcal{G})) & \longrightarrow & \zeta(\mathcal{Q}^F(\mathcal{G}))/\zeta(\mathcal{Q}^\mathcal{O}(\mathcal{G})) & \longrightarrow & 0,
\end{array}
\]

with exact rows.
Proof. Apart from the injectivity of \( \iota \), the existence of the top row and its exactness is (3.4). The exactness of the bottom row is tautological. Since \( \Lambda^O(\mathcal{G}) \) is a direct product of matrix rings over commutative local rings, [CR87, (45.12)] and a Morita equivalence argument show that the left vertical map is an isomorphism. Moreover, an extension of scalars argument shows that \( Q^F(\mathcal{G}) \) is direct product of matrix rings over fields, and so the middle vertical map is also an isomorphism. The left square commutes since the reduced norm / determinant map is compatible with extensions of scalars. The left and middle vertical isomorphisms induce the right vertical isomorphism, and so the right square commutes. Finally, commutativity of the diagram shows that \( \iota \) is injective. \( \square \)

10. Further algebraic results and the proof of Corollary 1.2

In this section, we begin by proving purely algebraic results on the vanishing of \( SK_1(\mathcal{G}) \) and on the injectivity of certain products of maps over subquotients of \( \mathcal{G} \). By combining these results with the functorial properties of the EIMC, we then show that Theorem 1.1 implies Corollary 1.2. Some results in this section are stated for all primes \( p \) and others are only stated for odd primes \( p \); those in the latter case ultimately rely on [RW05] where it is a standing hypothesis that \( p \) is odd.

10.1. \( F \)-\( q \)-elementary groups. Let \( q \) be a prime. A finite group is said to be \( q \)-hyperelementary if it is of the form \( C_n \rtimes Q \), with \( Q \) a \( q \)-group and \( C_n \) a cyclic group of order \( n \) such that \( q \nmid n \). Let \( F \) be a field of characteristic 0. A \( q \)-hyperelementary group \( C_n \rtimes Q \) is called \( F \)-\( q \)-elementary if

\[
\text{Im}(Q \to \text{Aut}(C_n) \cong (\mathbb{Z}/n\mathbb{Z})^\times) \subseteq \text{Gal}(F(\zeta_n)/F).
\]

An \( F \)-elementary group is one that is \( F \)-\( q \)-elementary for some prime \( q \). A finite group is said to be \( q \)-elementary if it is of the form \( C_n \rtimes Q \) with \( q \nmid n \) and \( Q \) a \( q \)-group.

Now let \( F/\mathbb{Q}_p \) be a finite extension and let \( \mathcal{G} \) be a one-dimensional compact \( p \)-adic Lie group. Let \( \Gamma_0 \simeq \mathbb{Z}_p \) be an open central subgroup of \( \mathcal{G} \). Then \( \mathcal{G} \) is said to be:

- \( F \)-\( q \)-elementary if there is a choice of \( \Gamma_0 \) such that \( \mathcal{G}/\Gamma_0 \) is \( F \)-\( q \)-elementary;
- \( F \)-elementary if it is \( F \)-\( q \)-elementary for some prime \( q \);
- \( q \)-elementary if there is a choice of \( \Gamma_0 \) such that \( \mathcal{G}/\Gamma_0 \) is \( q \)-elementary;
- \( \xi \)-elementary if it is \( q \)-elementary for some prime \( q \).

Lemma 10.1. In the case \( F = \mathbb{Q}_p \) the definition of \( F \)-\( q \)-elementary given above is equivalent to the corresponding definitions of [RW05, §2] (\( p = q \)) and [RW05, §3] (\( p \neq q \)).

Proof. Let \( \mathcal{G} \) be a one-dimensional compact \( p \)-adic Lie group. It is clear that if \( \mathcal{G} \) satisfies the definitions of Ritter and Weiss given in [RW05, §2, §3], then it satisfies the definition given above. The converse is given by [RW05, Lemma 4] in the case \( p \neq q \) and by the following calculation when \( p = q \). Suppose more generally that \( F/\mathbb{Q}_p \) is a finite extension and that we have a short exact sequence

\[
0 \to \Gamma_0 \to \mathcal{G} \to C_n \rtimes Q \to 0,
\]

where \( C_n \rtimes Q \) is \( F \)-\( p \)-elementary. Let \( s \in \mathcal{G} \) be a pre-image of a generator of \( C_n \). Since \( n \) and \( p \) are coprime, we may multiply \( s \) by a suitable element in \( \Gamma_0 \) to obtain an element of order \( n \). Thus we can and do assume without loss of generality that \( s \) itself has order \( n \). Let \( \mathcal{P} \subseteq \mathcal{G} \) be the pre-image of \( Q \). Then \( \mathcal{P} \) is a \( \text{pro-}p \)-group and \( \mathcal{G} \simeq C_n \rtimes \mathcal{P} \), where \( C_n \) is generated by \( s \). Since \( \Gamma_0 \) is central in \( \mathcal{G} \), the action of \( \mathcal{P} \) on \( C_n \) factors through \( \mathcal{P} \to Q \to \text{Aut}(C_n) \) and thus has image in \( \text{Gal}(F(\zeta_n)/F) \). \( \square \)
10.2. **The kernel of the reduced norm map.** Let \( p \) be a prime and let \( \mathcal{G} \) be a one-dimensional compact \( p \)-adic Lie group. Let \( F/\mathbb{Q}_p \) be a finite extension. Define

\[
SK_1(Q^F(\mathcal{G})) = \ker(nr : K_1(Q^F(\mathcal{G})) \to \zeta(Q^F(\mathcal{G}))^\times).
\]

**Proposition 10.2.** Let \( p \) be an odd prime and let \( \mathcal{G} \) be a one-dimensional compact \( p \)-adic Lie group. Then \( SK_1(Q(\mathcal{G})) = 0 \) if \( SK_1(Q(\mathcal{H})) = 0 \) for all open \( \mathbb{Q}_p \)-\( p \)-elementary subgroups \( \mathcal{H} \) of \( \mathcal{G} \).

**Proof.** By [RW05, Corollary on p. 167] we have that \( \ker(nr : K_1(Q^F(\mathcal{G})) \to \zeta(Q^F(\mathcal{G}))^\times) \) is injective. If we further assume that \( \mathcal{H} \) is an open \( \mathbb{Q}_p \)-\( p \)-elementary subgroup then \( SK_1(Q(\mathcal{H})) = 0 \) by a result of Lau [Lau12, Theorem 2]. \( \square \)

**Corollary 10.3.** If \( p \) is an odd prime and \( \mathcal{G} \) is a one-dimensional compact \( p \)-adic Lie group with an abelian Sylow \( p \)-subgroup then \( SK_1(Q(\mathcal{G})) = 0 \).

**Proof.** By Proposition 10.2 it suffices to show that \( SK_1(Q(\mathcal{H})) = 0 \) for all open \( \mathbb{Q}_p \)-\( p \)-elementary subgroups \( \mathcal{H} \) of \( \mathcal{G} \). Let \( \mathcal{H} \) be such a subgroup. Then \( \mathcal{H} = \langle s \rangle \times \mathcal{U} \) where \( \langle s \rangle \) is a finite cyclic subgroup of order prime to \( p \) and \( \mathcal{U} \) is an open pro-\( p \) subgroup. Moreover, \( \mathcal{U} \) must be abelian by the hypothesis on \( \mathcal{G} \) and so the commutator subgroup \( \mathcal{H}' \) of \( \mathcal{H} \) is necessarily a subgroup of \( \langle s \rangle \). Hence \( p \nmid |\mathcal{H}'| \) and so the reduced norm map \( nr : K_1(Q(\mathcal{H})) \to \zeta(Q(\mathcal{H}))^\times \) is an isomorphism by Corollary 9.2 (with \( F = \mathbb{Q}_p \)). In particular, \( SK_1(Q(\mathcal{H})) = 0 \). \( \square \)

**Remark 10.4.** As noted in [RW04, Remark E] (also see [Bur15, Remark 3.5]), a conjecture of Suslin implies that \( SK_1(Q(\mathcal{G})) \) in fact always vanishes.

10.3. **Products of maps over subquotients of \( \mathcal{G} \).** We let \( nr(K_1(\Lambda(\mathcal{G}))) \) denote the image of \( K_1(\Lambda(\mathcal{G})) \) under the composition of the two maps \( K_1(\Lambda(\mathcal{G})) \to K_1(Q(\mathcal{G})) \) and \( nr : K_1(Q(\mathcal{G})) \to \zeta(Q(\mathcal{G}))^\times \). The purpose of this subsection is to prove the following result.

**Theorem 10.5.** Let \( p \) be an odd prime and let \( \mathcal{G} = H \times \Gamma \) be a one-dimensional compact \( p \)-adic Lie group. Let \( \mathcal{C} \) be the collection of all \( p \)-elementary subquotients of \( \mathcal{G} \) of the form \( \mathcal{U}/N \), where \( \mathcal{U} \) is open in \( \mathcal{G} \) and \( N \) is a finite normal subgroup of \( \mathcal{U} \). Then the product of maps

\[
\zeta(Q(\mathcal{G}))^\times/\text{nr}(K_1(\Lambda(\mathcal{G}))) \to \prod_{\mathcal{H} \in \mathcal{C}} \zeta(Q(\mathcal{H}))^\times/\text{nr}(K_1(\Lambda(\mathcal{H})))
\]

is injective. If we further assume that \( SK_1(Q(\mathcal{G})) = 0 \), then the product of maps

\[
K_0(\Lambda(\mathcal{G}), Q(\mathcal{G})) \to \prod_{\mathcal{H} \in \mathcal{C}} K_0(\Lambda(\mathcal{H}), Q(\mathcal{H}))
\]

is also injective.

We shall first prove several auxiliary and intermediate results which may be of interest on their own right.

**Lemma 10.6.** Let \( p \) be a prime and let \( \mathcal{G} \) be a one-dimensional compact \( p \)-adic Lie group. Let \( F/\mathbb{Q}_p \) be a finite extension with ring of integers \( \mathcal{O} \). Then there exists a commutative
Let $F$ be the ring of integers of $\mathbb{F}$. Let $\Lambda$ be a one-dimensional compact $p$-adic Lie group. Let $F/\mathbb{Q}_p$ be a finite extension that is at most tamely ramified and let $\mathcal{O}$ be the ring of integers of $F$. Then the natural map

$$\zeta(\mathcal{O}(\mathcal{G}))^\times/\text{nr}(K_1(\Lambda(\mathcal{G}))) \to \zeta(F(\mathcal{G}))^\times/\text{nr}(K_1(\Lambda^\mathcal{O}(\mathcal{G})))$$

is injective. If we further assume that $SK_1(\mathcal{O}(\mathcal{G})) = 0$, then the extension of scalars map

$$K_0(\Lambda(\mathcal{G}), \mathcal{O}(\mathcal{G})) \to K_0(\Lambda^\mathcal{O}(\mathcal{G}), F(\mathcal{G}))$$

is also injective.

Proof. By enlarging $F$ if necessary, we can and do assume that $F/\mathbb{Q}_p$ is Galois. The first claim follows from the equalities

$$\zeta(\mathcal{O}(\mathcal{G}))^\times \cap \text{nr}(K_1(\Lambda^\mathcal{O}(\mathcal{G}))) = \text{nr}(K_1(\Lambda^\mathcal{O}(\mathcal{G})))^{\text{Gal}(F/\mathbb{Q}_p)} = \text{nr}(K_1(\Lambda(\mathcal{G}))),$$

where the last equality is [IV12, Theorem 2.12]. We have a commutative diagram

$$\begin{array}{ccc}
K_0(\Lambda(\mathcal{G}), \mathcal{O}(\mathcal{G})) & \longrightarrow & \zeta(\mathcal{O}(\mathcal{G}))^\times/\text{nr}(K_1(\Lambda(\mathcal{G}))) \\
\downarrow & & \downarrow \\
K_0(\Lambda^\mathcal{O}(\mathcal{G}), F(\mathcal{G})) & \longrightarrow & \zeta(F(\mathcal{G}))^\times/\text{nr}(K_1(\Lambda^\mathcal{O}(\mathcal{G}))),
\end{array}$$

where the existence of the horizontal maps follows from Lemma 10.6. If $SK_1(\mathcal{O}(\mathcal{G})) = 0$ then the top horizontal map is injective by Lemma 10.6, and so the second claim now follows from the commutativity of the diagram. }

Lemma 10.8. Let $p$ be a prime and let $\mathcal{G}$ be a one-dimensional compact $p$-adic Lie group. Let $F/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$. Then the natural map

$$\zeta(\mathcal{O}(\mathcal{G}))^\times/\zeta(\Lambda(\mathcal{G}))^\times \to \zeta(F(\mathcal{G}))^\times/\zeta(\Lambda^\mathcal{O}(\mathcal{G}))^\times$$

is injective.

Proof. Write $\mathcal{G} = H \times \Gamma$ where $H$ is finite and $\Gamma \simeq \mathbb{Z}_p$. Let $\Gamma_0$ be an open subgroup of $\Gamma$ that is central in $\mathcal{G}$. Let $\mathcal{O}$ be an open subgroup of $\Gamma$ that is central in $\mathcal{G}$. Let $\mathcal{O} = \mathcal{O}_0[\Gamma]$. Since $\zeta(\Lambda(\mathcal{G}))$ and $\zeta(\Lambda^\mathcal{O}(\mathcal{G}))$ are both $\mathcal{O}$-orders, all of their elements are integral over $\mathcal{O}$ by [Rei03, Theorem 8.6]. Thus $\zeta(\Lambda^\mathcal{O}(\mathcal{G}))^\times \cap \zeta(\Lambda(\mathcal{G}))^\times = \zeta(\Lambda(\mathcal{G}))^\times$ by [Swa83, Lemma 9.7]. Hence we have

$$\zeta(\Lambda(\mathcal{G}))^\times \subseteq \zeta(\mathcal{O}(\mathcal{G}))^\times \cap \zeta(\Lambda^\mathcal{O}(\mathcal{G}))^\times \subseteq \zeta(\Lambda(\mathcal{G})) \cap \zeta(\Lambda^\mathcal{O}(\mathcal{G}))^\times = \zeta(\Lambda(\mathcal{G}))^\times.$$ 

Therefore $\zeta(\mathcal{O}(\mathcal{G}))^\times \cap \zeta(\Lambda^\mathcal{O}(\mathcal{G}))^\times = \zeta(\Lambda(\mathcal{G}))^\times$, which gives the desired result.

In the results that follow, the quotient and restriction maps on certain quotients of $\zeta(F(\mathcal{G}))^\times$ are induced by those defined in §3.6.
Proposition 10.9. Let $p$ be a prime and let $G = H \times \Delta$ where $H$ is a one-dimensional compact $p$-adic Lie group such that $p \nmid |H'|$ and $\Delta$ is a finite group with $p \nmid |\Delta|$. Let $C(\Delta)$ be the collection of cyclic subquotients of $\Delta$. Then the products of maps

\[
\begin{align*}
\zeta(Q(G))^x/\zeta(\Lambda(G))^x &\rightarrow \prod_{C \in C(\Delta)} \zeta(Q(H \times C))^x/\zeta(\Lambda(G \times C))^x, \\
\zeta(Q(G))^x/\text{nr}(K_1(\Lambda(G))) &\rightarrow \prod_{C \in C(\Delta)} \zeta(Q(H \times C))^x/\text{nr}(K_1(\Lambda(G \times C))),
\end{align*}
\]

and

\[
K_0(\Lambda(G), Q(G)) \rightarrow \prod_{C \in C(\Delta)} K_0(\Lambda(H \times C), Q(H \times C))
\]

are all injective.

Proof. The hypotheses imply that $p \nmid |G'|$. Hence Corollary 9.2 implies that injectivity of the second and third displayed maps follows from that of the first displayed map.

Set $d := |\Delta|$. Let $F = Q_p(\zeta_d)$ and let $O$ be the ring of integers of $F$. Then $F/Q_p$ is a finite unramified extension over which every representation of every subgroup of $\Delta$ can be realised. Then there is a canonical decomposition $\zeta(O[\Delta]) \cong \prod_{\psi \in \text{Irr}_{Q_p}(\Delta)} O$, where the sum runs over all $\psi \in \text{Irr}_{Q_p}(\Delta)$. This decomposition induces an isomorphism

\[
\zeta(Q(F(G))^x/\zeta(\Lambda^O(G))^x \cong \prod_{\psi \in \text{Irr}_{Q_p}(\Delta)} \zeta(Q(F(H))^x/\zeta(\Lambda^O(H))^x.
\]

Similar observations hold for the quotients $\zeta(Q(F(H \times C))^x/\zeta(\Lambda^O(H \times C))^x$ for each $C \in C(\Delta)$. Moreover, we have a commutative diagram

\[
\begin{array}{ccc}
\zeta(Q(G))^x/\zeta(\Lambda(G))^x & \rightarrow & \prod_{C \in C(\Delta)} \zeta(Q(H \times C))^x/\zeta(\Lambda(G \times C))^x \\
\downarrow & & \downarrow \\
\zeta(Q(F(G))^x/\zeta(\Lambda^O(G))^x & \rightarrow & \prod_{C \in C(\Delta)} \zeta(Q(F(H \times C))^x/\zeta(\Lambda^O(H \times C))^x
\end{array}
\]

where the vertical extension of scalars maps are injective by Lemma 10.8. Thus it suffices to show that the bottom horizontal map is injective.

Now let $f$ be an arbitrary element in $\zeta(Q(F(G))^x/\zeta(\Lambda^O(G))^x$. Write $f = (f_\psi)_\psi$ with $f_\psi \in \zeta(Q(F(H))^x/\zeta(\Lambda^O(H))^x$. Let $(f_C)_{C \in C(\Delta)}$ be the image of $f$ under the bottom horizontal map in the diagram. For each $C \in C(\Delta)$ we write $f_C = (f_{C,\lambda})_\lambda$, where $\lambda$ runs through the irreducible characters of $C$. If $C = U/N$ for a subgroup $U$ of $\Delta$ and a normal subgroup $N$ of $U$, then explicitly we have

\[
f_{C,\lambda} = \prod_{\psi} f_\psi^{(\psi,\text{ind}_{U}^{H}\text{ind}_{N}^{U}\lambda)}.
\]

Now suppose that $f$ belongs to the kernel of the bottom map. Then we have that $f_{C,\lambda} \in \zeta(\Lambda^O(H))^x$ for each $C \in C(\Delta)$, $\lambda \in \text{Irr}_{Q_p}(C)$. Now fix an irreducible character $\psi \in \text{Irr}_{Q_p}(\Delta)$. By Brauer’s induction theorem we may write $\psi$ as a finite sum $\psi = \sum_j z_j \text{ind}_{C_j}^{\Delta} \text{ind}_{C_j}^{H} \lambda_j$ where each $\lambda_j$ is an irreducible character of a cyclic subquotient $C_j =$
Let \( p \) be an odd prime and let \( G \) be a one-dimensional compact \( p \)-adic Lie group. Let \( F/\mathbb{Q}_p \) be a finite extension with ring of integers \( \mathcal{O} \). Let \( C \) be the collection of all open \( F \)-elementary subgroups of \( G \). Then both products of maps

\[
\zeta(Q^F(G))^\times / \text{nr}(K_1(\Lambda^O(G))) \xrightarrow{\prod \text{res}_H^\mathcal{O}} \prod_{\mathcal{H} \in C} \zeta(Q^F(\mathcal{H}))^\times / \text{nr}(K_1(\Lambda^O(\mathcal{H})))
\]

and

\[
K_0(\Lambda^O(G), Q^F(G)) \xrightarrow{\prod \text{res}_H^\mathcal{O}} \prod_{\mathcal{H} \in C} K_0(\Lambda^O(\mathcal{H}), Q^F(\mathcal{H}))
\]

are injective.

**Proof.** For an open subgroup \( U \) of \( G \), we denote by \( R_F(U) \) the ring of all characters of finite dimensional \( F \)-representations of \( U \) with open kernel. By [RW05, Lemma 7] the groups \( K_1(\Lambda^O(-)) \) and \( K_1(\mathcal{Q}(-)) \) are Frobenius modules over the Frobenius functor \( U \mapsto R_{Q_p}(U) \) (note that the result for \( K_1(\mathcal{Q}(-)) \) is not explicitly stated, but the same proof works, and it is actually used in the subsequent corollary). The same argument shows that \( K_1(\Lambda^O(-)) \) and \( K_1(Q^F(-)) \) are Frobenius modules over the Frobenius functor \( U \mapsto R_F(U) \). The natural map \( K_1(\Lambda^O(-)) \to K_1(Q^F(-)) \) is a morphism of Frobenius modules and thus its cokernel \( K_0(\Lambda^O(-), Q^F(-)) \) is also a Frobenius module over the Frobenius functor \( U \mapsto R_F(U) \). Likewise, the map \( \text{Det} : K_1(\Lambda(-)) \to \text{Hom}_{\mathcal{Q}_p}(R_p(-), Q^F(\Gamma)^\times) \) is a morphism of Frobenius modules by [RW05, Lemma 7] and the same argument works with \( K_1(\Lambda(-)) \) replaced with \( K_1(\Lambda^O(-)) \). Hence the cokernel \( \zeta(Q^F(-))^\times / \text{nr}(K_1(\Lambda^O(-))) \) is a Frobenius module over the Frobenius functor \( U \mapsto R_F(U) \). We now conclude as in the proof of [RW05, Corollary, p. 167]: Let \( 1_G \) be the trivial character of \( G \). Then the Witt–Bermann induction theorem [CR81, Theorem 21.6] implies that \( 1_G \) may be written as a finite sum

\[
1_G = \sum_{\mathcal{H} \in C} z_H \text{ind}_{\mathcal{H}}^G \xi_H
\]

with \( z_H \in \mathbb{Z} \) and \( \xi_H \in \text{Irr}_F(\mathcal{H}) \). Now let \( x \) be either in \( \zeta(Q^F(G))^\times / \text{nr}(K_1(\Lambda^O(G))) \) or in \( K_0(\Lambda^O(G), Q^F(G)) \) and suppose that \( x \) is in the kernel of \( \prod \text{res}_H^\mathcal{O} \). Then we see that

\[
x = 1_G \cdot x = \sum_{\mathcal{H} \in C} z_H (\text{ind}_{\mathcal{H}}^G \xi_H) \cdot x = \sum_{\mathcal{H} \in C} z_H \text{ind}_{\mathcal{H}}^G (\xi_H \cdot \text{res}_H^\mathcal{O} x) = 0
\]

as desired. \( \square \)

**Remark 10.11.** Let \( C' \) be the finite set of those \( \mathcal{H} \in C \) for which \( z_H \neq 0 \) in (10.2). It is then clear from the proof that it indeed suffices to consider the collection of all \( \mathcal{H} \in C' \) in the statement of Proposition 10.10.
Corollary 10.12. Let $p$ be an odd prime and let $\mathcal{G}$ be a one-dimensional compact $p$-adic Lie group. Let $\mathcal{C}$ be the collection of all open elementary subgroups of $\mathcal{G}$. Then the product of maps

$$\zeta(Q(\mathcal{G}))^*/\text{nr}(K_1(\Lambda(\mathcal{G}))) \prod_{n \in \mathcal{C}} \zeta(Q(\mathcal{H}))^*/\text{nr}(K_1(\Lambda(\mathcal{H})))$$

is injective. If we further assume that $SK_1(Q(\mathcal{G})) = 0$, then the product of maps

$$K_0(\Lambda(\mathcal{G}), Q(\mathcal{G})) \prod_{n \in \mathcal{C}} K_0(\Lambda(\mathcal{H}), Q(\mathcal{H}))$$

is also injective.

Proof. Write $\mathcal{G} = H \times \Gamma$ where $H$ is finite and $\Gamma \simeq \mathbb{Z}_p$. Write $|H| = p^tk$ for integers $t$ and $k$ such that $t \geq 0$ and $p \nmid k$. Then $F := \mathbb{Q}_p(\zeta_{pk})$ is a finite tamely ramified extension of $\mathbb{Q}_p$. We now repeat an argument given in the proof of [GRW99, Proposition 9] to show that every $F$-elementary subgroup of any finite quotient of $\mathcal{G}$ is in fact elementary. Let $q$ be a prime and let $C_q \times Q$ be an $F$-q-elementary finite quotient of $\mathcal{G}$. Write $n = p^m$ for integers $s$ and $m$ such that $s \geq 0$ and $p \nmid k$. Note that $m$ must divide $k$. Since both $\zeta_p$ and $\zeta_m$ lie in $F$, the Galois group $\text{Gal}(F(\zeta_n)/F)$ has $p$-power order. Thus if $p \neq q$ then any homomorphism $Q \to \text{Gal}(F(\zeta_n)/F)$ must be trivial. If $q = p$ then $s = 0$ and so the extension $F(\zeta_n)/F$ is trivial, giving the same result.

Now Proposition 10.10 and Lemma 10.7 imply the first claim. The second claim follows from Lemma 10.6. \qed

Proof of Theorem 10.5. This follows from Corollary 10.12 and Proposition 10.9. We only have to observe that for a prime $q \neq p$ a $q$-elementary open subgroup of $\mathcal{G}$ is of the form $\Gamma \times C \times \Delta$, where $\Gamma \simeq \mathbb{Z}_p$, $C$ is finite cyclic, and $\Delta$ is a finite $q$-group (see [RW05, Lemma 4]). We may therefore apply Proposition 10.9 with $\mathcal{H} = \Gamma \times C$. The groups $\mathcal{H} \times H$ are $p$-elementary (and abelian) for all cyclic subquotients $H$ of $\Delta$. \qed

10.4. Application to the EIMC. We give an easy reformulation of the EIMC without its uniqueness statement.

Lemma 10.13. Let $p$ be an odd prime and let $\mathcal{L}/K$ be an admissible one-dimensional $p$-adic Lie extension of a totally real number field $K$. Let $\mathcal{G} = \text{Gal}(\mathcal{L}/K)$ and let $S$ be a finite set of places of $K$ containing $S_{\text{ram}} \cup S_{\infty}$. Choose any $\zeta_S \in K_1(Q(\mathcal{G}))$ such that $\partial(\zeta_S) = -[C_S^*(\mathcal{L}/K)]$. Then the EIMC holds for $\mathcal{L}/K$ if and only if

$$\text{nr}(\zeta_S) \equiv \Phi_S \mod \text{nr}(K_1(\Lambda(\mathcal{G}))).$$

Proof. This is an easy consequence of Lemma 10.6 in the case $F = \mathbb{Q}_p$. \qed

We are now ready to prove the main result of this section.

Theorem 10.14. Let $p$ be an odd prime and let $K$ be a totally real number field. Let $\mathcal{L}/K$ be an admissible one-dimensional $p$-adic Lie extension. Then the EIMC holds for $\mathcal{L}/K$ if and only if it holds for all intermediate admissible extensions with $p$-elementary Galois group.

Proof. If the EIMC holds for $\mathcal{L}/K$ then it holds for every admissible intermediate extension by Lemma 4.10. Conversely, Propositions 4.2 and 4.3, Lemma 10.6 in the case $F = \mathbb{Q}_p$, Lemma 10.13 and Theorem 10.5 show that the EIMC for $\mathcal{L}/K$ holds if it holds for all intermediate extensions with $p$-elementary Galois group. \qed
Remark 10.15. If the extension $L/K$ satisfies the $\mu = 0$ hypothesis, then [RW05, Theorem A] shows that Theorem 10.14 (i) recovers [RW05, Theorem C] (which itself relies on the vanishing of $\mu$).

Proof of Corollary 1.2. Let $G = \text{Gal}(L/K)$. Since $G$ has an abelian Sylow $p$-subgroup, each $p$-elementary subquotient of $G$ is also abelian. Hence the EIMC for $L/K$ holds by the combination of Theorems 1.1 and 10.14. Moreover, $SK_1(Q(G)) = 0$ by Corollary 10.3 and so we also have uniqueness (see Remark 4.8). \hfill \Box

11. THE ETNC AT NEGATIVE INTEGERS AND THE COATES–SINNOTT CONJECTURE

The equivariant Tamagawa number conjecture (ETNC) has been formulated by Burns and Flach [BF01] in vast generality. We will only consider the case of Tate motives. So let $L/K$ be a finite Galois extension of number fields with Galois group $G$ and let $r$ be an integer. We regard $h^0(\text{Spec}(L))(r)$ as a motive defined over $K$ and with coefficients in the semisimple algebra $Q[G]$. The ETNC for the pair $(h^0(\text{Spec}(L))(r), Z[G])$ simply asserts that a certain canonical element $T\Omega(L/K, Z[G], r) \in K_0(Z[G], \mathbb{R})$ vanishes.

Now we assume that $L/K$ is a CM extension. By this we mean that $L$ is a totally real number field and that $L$ is a totally complex extension of $K$ such that complex conjugation induces a unique automorphism $j \in G$. Then $j$ is central in $G$ and we denote the maximal totally real subfield of $L$ by $L^+$. Note that $L/L^+$ is a quadratic extension and its Galois group is generated by $j$. Away from its 2-primary part, the ETNC then naturally decomposes into a plus and a minus part.

For each $r \in \mathbb{Z}$ we define a central idempotent $e_r := \frac{1 + (-1)^{\text{ord}_2 r}}{2}$ in $Z[\frac{1}{2}][G]$. The ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r Z[\frac{1}{2}][G])$ then likewise asserts that a certain canonical element $T\Omega(L/K, e_r Z[\frac{1}{2}][G], r) \in K_0(e_r Z[\frac{1}{2}][G], \mathbb{R})$ vanishes. This corresponds to the plus or minus part (away from 2) if $r$ is odd or even.

If $r$ is a negative integer, then a result of Siegel [Sie70] implies that $T\Omega(L/K, e_r Z[\frac{1}{2}][G], r)$ actually belongs to the subgroup

$$K_0(e_r Z[\frac{1}{2}][G], \mathbb{Q}) \cong \bigoplus_{p \text{odd}} K_0(e_r Z_p[G], \mathbb{Q}_p)$$

and we say that the $p$-part of the ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r Z[\frac{1}{2}][G])$ holds if its image in $K_0(e_r Z_p[G], \mathbb{Q}_p)$ vanishes.

Theorem 11.1. Let $p$ be an odd prime. Let $L/K$ be a finite Galois CM extension of number fields with Galois group $G$. Then the following hold for every negative integer $r$.

(i) The element $T\Omega(L/K, e_r Z[\frac{1}{2}][G], r)$ belongs to $K_0(e_r Z[\frac{1}{2}][G], \mathbb{Q})_{\text{tors}}$.

(ii) Assume that the extension $L(\zeta_p)^\infty/K$ satisfies the $\mu = 0$ hypothesis if $p$ divides $|G|$. Then the $p$-part of the ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r Z[\frac{1}{2}][G])$ holds.

Proof. Part (ii) has been shown by Burns [Bur15, Corollary 2.10]. If $L(\zeta_p)^\infty/K$ satisfies the $\mu = 0$ hypothesis (whether or not $p$ divides $|G|$) there is an independent proof due to the second named author [Nic13, Corollary 5.11]. By a general induction argument [Nic11, Proposition 6.1(iii)] (ii) implies (i) (if $r$ is odd see also [Nic11, Corollary 6.2]). \hfill \Box

We now remove the hypothesis that $\mu$ vanishes whenever $G$ has an abelian Sylow $p$-subgroup. This is Theorem 1.3 from the introduction.

Theorem 11.2. Let $p$ be an odd prime. Let $L/K$ be a finite Galois CM extension of number fields with Galois group $G$. Suppose that $G$ has an abelian Sylow $p$-subgroup. Then
for each negative integer $r$ the $p$-part of the ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r\mathbb{Z}_{\frac{1}{2}}[G])$ holds.

Proof. Let $S$ and $T$ be two finite non-empty sets of places of $K$ such that $S$ contains $S_{\text{ram}} \cup S_{\infty}$ and such that $T \cap S$ is empty. We assume in addition that all $p$-adic places lie in $S$. We define a complex of $e_r\mathbb{Z}_p[G]$-modules

$$R\Gamma_T(\mathcal{O}_{K,S}, e_r\mathbb{Z}_p[G]^\#(1-r)) := \text{cone}(R\Gamma(\text{Spec}(\mathcal{O}_{K,S}), e_r\mathbb{Z}_p[G]^\#(1-r)))$$

$$\rightarrow \bigoplus_{v \in T} R\Gamma(\text{Spec}(K(v)), e_r\mathbb{Z}_p[G]^\#(1-r)))[-1],$$

where $K(v)$ denotes the residue field of $K$ at $v$. By [Nic13, Theorem 5.10] this complex is acyclic outside degree 2 and the only non-vanishing cohomology group, which we denote by $H^2_T(\mathcal{O}_{K,S}, e_r\mathbb{Z}_p[G]^\#(1-r))$, is cohomologically trivial. Moreover, the ETNC for the pair $(h^0(\text{Spec}(L))(r), e_r\mathbb{Z}_{\frac{1}{2}}[G])$ holds if and only if $\Theta_{S,T}(r)$ is a generator of the (non-commutative) Fitting invariant of this $e_r\mathbb{Z}_p[G]$-module.

We now can either work with non-commutative Fitting invariants or we can apply [GRW99, Proposition 9] in combination with Theorem 11.1 (i) to reduce to abelian extensions. We choose the latter option so that the result follows from Lemma 11.3 below.

The following result is a strengthening of the ‘strong Coates–Sinnott conjecture’ [Nic13, Conjecture 5.1] in the case of abelian CM extensions.

**Lemma 11.3.** Let $p$ be an odd prime. Let $L/K$ be a finite abelian CM extension of number fields with Galois group $G$. Let $S$ and $T$ be two finite non-empty sets of places of $K$ such that $S$ contains $S_{\text{ram}} \cup S_{\infty} \cup S_p$ and $S \cap T = \emptyset$. Then for each negative integer $r$ we have

$$\text{Fitt}_{e_r\mathbb{Z}_p[G]}(H^2_T(\mathcal{O}_{K,S}, e_r\mathbb{Z}_p[G]^\#(1-r))) = \Theta_{S,T}(r)e_r\mathbb{Z}_p[G].$$

Proof. We first observe that it suffices to show that $\Theta_{S,T}(r)$ is contained in the Fitting ideal by [Nic13, Theorem 5.10]. Hence we may and do assume that $L$ contains a primitive $p$th root of unity by [Nic13, Proposition 5.5]. Let $L_{\infty}$ and $K_{\infty}$ be the cyclotomic $Z_p$-extensions of $L$ and $K$, respectively. Set $G := \text{Gal}(L_{\infty}/K)$. Then $G \simeq H \times \Gamma$ where $H = \text{Gal}(L_{\infty}/K_{\infty})$ and $\Gamma \simeq Z_p$. Moreover, we have that $\Lambda(G) = R[H]$ where $R := \Lambda(\Gamma)$. For each integer $n$ we now define a complex of $e_n\Lambda(G)$-modules

$$R\Gamma_T(\mathcal{O}_{K,S}, e_n\Lambda(G)^\#(1-n)) := \text{cone}(R\Gamma(\text{Spec}(\mathcal{O}_{K,S}), e_n\Lambda(G)^\#(1-n)))$$

$$\rightarrow \bigoplus_{v \in T} R\Gamma(\text{Spec}(K(v)), e_n\Lambda(G)^\#(1-n)))[-1],$$

In the case $n = 0$ this complex has been studied by Burns [Bur20, §5.3.1]. It is acyclic outside degree 2 and the second cohomology group is of projective dimension at most 1 by [Bur20, Proposition 5.5]. By [Nic19b, Remark 4.10] one has indeed that

$$H^2_T(\mathcal{O}_{K,S}, e_0\Lambda(G)^\#(1)) \cong E^1(Y_{S,T}^+(1)).$$

Taking the $-r$-fold Tate twist we obtain

$$H^2_T(\mathcal{O}_{K,S}, e_r\Lambda(G)^\#(1-r)) \cong E^1(Y_{S,T}^+(1))(-r).$$

As the main conjecture holds for $L_\infty^+/K$ by Theorem 8.5, Proposition 8.3 implies that the Fitting ideal of $Y_{S,T}^+(1)$ is generated by $\Psi_{S,T}$. Then $\Psi_{S,T}^\#$ generates the Fitting
ideal of $E^1(Y^T_S(-1))$ by Lemma 6.5 and likewise $t^*_{cyc}(\Psi^\#_{S,T})$ generates the fitting ideal of $E^1(Y^T_S(-1)(-r))$. We have shown that

\[(11.1) \quad \text{Fitt}_{e_r,\Lambda(G)}(H^2_F(O_{K,S}, e_r\Lambda(G)^\#(1-r))) = t^*_{cyc}(\Psi^\#_{S,T})e_r\Lambda(G).\]

Since we have natural isomorphisms in $\mathcal{D}(\mathbb{Z}_p[G])$ of the form

$$\mathbb{Z}_p[G] \otimes_{\Lambda(G)}^L \Gamma(Spec(O_{K,S}), \Lambda(G)^\#(1-r)) \simeq \mathbb{R}\Gamma(Spec(O_{K,S}), \mathbb{Z}_p[G]^\#(1-r))$$

and likewise

$$\mathbb{Z}_p[G] \otimes_{\Lambda(G)}^L \Gamma(Spec(K(v)), \Lambda(G)^\#(1-r)) \simeq \mathbb{R}\Gamma(Spec(K(v)), \mathbb{Z}_p[G]^\#(1-r))$$

for each $v \in T$ by [FK06, Proposition 1.6.5], we have a natural isomorphism in $\mathcal{D}(e_r\mathbb{Z}_p[G])$ of the form

$$e_r\mathbb{Z}_p[G] \otimes_{e_r\Lambda(G)}^L \Gamma_T(O_{K,S}, e_r\Lambda(G)^\#(1-r)) \simeq \mathbb{R}\Gamma_T(O_{K,S}, e_r\mathbb{Z}_p[G]^\#(1-r)).$$

However, both complexes in this formula are acyclic outside degree 2 so that we actually have an isomorphism of $e_r\mathbb{Z}_p[G]$-modules

$$H^2_F(O_{K,S}, e_r\Lambda(G)^\#(1-r))_{\Gamma_L} \cong H^2_F(O_{K,S}, e_r\mathbb{Z}_p[G]^\#(1-r)),$$

where $\Gamma_L := \text{Gal}(L_{\infty}/L)$. Let $\text{aug} : \Lambda(G) \to \mathbb{Z}_p[G]$ be the canonical projection map. Then (11.1) and Lemma 5.2 imply that $\text{aug}(t^*_{cyc}(\Psi^\#_{S,T}))$ generates the fitting ideal of $H^2_F(O_{K,S}, e_r\mathbb{Z}_p[G]^\#(1-r))$. As $\text{aug}(t^*_{cyc}(\Psi^\#_{S,T})) = \Theta_{S,T}(r)$ by [GP15, Lemma 5.14 (2)], we are done.

Now let $L/K$ be an arbitrary finite abelian Galois extension of number fields with Galois group $G$. For each integer $r$ we define an idempotent in $\mathbb{Z}[\frac{1}{2}][G]$ by

$$e_r := \begin{cases} \prod_{v \in S_\infty} \frac{1-(-1)^{j_v}}{2} & \text{if } K \text{ is totally real;} \\ 0 & \text{otherwise,} \end{cases}$$

where $j_v$ is the generator of the decomposition group $G_v$ for each $v \in S_\infty$. Note that this is compatible with the above definition of $e_r$ in the case of CM extensions.

We obtain the following refinement of Theorem 1.5.

**Corollary 11.4.** Let $L/K$ be a finite abelian Galois extension of number fields with Galois group $G$. Then one has an equality

\[(11.2) \quad \text{Ann}_{\mathbb{Z}[\frac{1}{2}][G]}(\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{Z} K_{1-2r}(O_{L,tors})\Theta_S(r) = e_r\text{Fitt}_{\mathbb{Z}[\frac{1}{2}][G]}(\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{Z} K_{-2r}(O_{L,S}))\]

for every finite set $S$ of places of $K$ containing $S_{\text{ram}} \cup S_{\infty}$. In particular, the Coates–Sinnott conjecture 1.4 holds away from 2, that is

$$\text{Ann}_{\mathbb{Z}[\frac{1}{2}][G]}(\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{Z} K_{1-2r}(O_{L,tors})\Theta_S(r) \subseteq \text{Ann}_{\mathbb{Z}[\frac{1}{2}][G]}(\mathbb{Z}[\frac{1}{2}] \otimes \mathbb{Z} K_{-2r}(O_{L,S})).$$

**Proof.** Fix an odd prime $p$. In order to verify the $p$-part of (11.2), we may and do assume that $S_p \subset S$ as the Euler factors at $v \in S_p$ are units in $\mathbb{Z}_p[G]$ by [GP15, Lemma 6.13]. Moreover, since the $p$-adic Chern class maps (1.2) are isomorphisms by the norm residue isomorphism theorem [Wei09], we may work with the étale cohomological version of (11.2) as in [GP15, §6] (where the corresponding claim is denoted by $\overline{CS}(L/K, S_p, 1-r)$). By [GP15, Lemma 6.14] we may assume that $K$ is totally real. Likewise, by [GP15, Lemmas...
6.15 and 6.16] we may assume that \( L \) is a CM extension of \( K \). We have an exact sequence of finite \( e_r \mathbb{Z}_p[G] \)-module (this follows easily from the definitions; see [Nic13, (21)])

\[
0 \longrightarrow H^1_{\text{ét}}(\text{Spec}(\mathcal{O}_{K,S}), e_r \mathbb{Z}_p[G]#(1-r)) \longrightarrow \bigoplus_{v \in T} H^1_{\text{ét}}(\text{Spec}(K(v)), e_r \mathbb{Z}_p[G]#(1-r))
\]

\[
\longrightarrow H^2_{\text{ét}}(\text{Spec}(\mathcal{O}_{K,S}), e_r \mathbb{Z}_p[G]#(1-r)) \longrightarrow H^1_{\text{ét}}(\text{Spec}(\mathcal{O}_K), e_r \mathbb{Z}_p[G]#(1-r)) \longrightarrow 0.
\]

Here, the middle two terms are finite cohomologically trivial \( G \)-modules and their Fitting ideals are generated by \( \delta_T(r) \) and \( \Theta_{S,T}(r) \) by [Nic13, Lemma 5.4] and Lemma 11.3, respectively. Since \( H^1_{\text{ét}}(\text{Spec}(\mathcal{O}_{K,S}), e_r \mathbb{Z}_p[G]#(1-r)) \) is finite cyclic, we have that

\[
\text{Fitt}_{e_r \mathbb{Z}_p[G]}(H^1_{\text{ét}}(\text{Spec}(\mathcal{O}_{K,S}), e_r \mathbb{Z}_p[G]#(1-r))^\vee)^# = \text{Ann}_{e_r \mathbb{Z}_p[G]}(H^1_{\text{ét}}(\text{Spec}(\mathcal{O}_{L,S}), \mathbb{Z}_p(1-r))_{\text{tors}}).
\]

Here, we have used Shapiro’s lemma for the last equality. The result now follows from Lemma 5.3.

\[\square\]

**REFERENCES**


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