Research seminar:
Higher Chern classes in Iwasawa theory

(after Bleher, Chinburg, Greenberg, Kakde, Pappas, Sharifi, Taylor)

Andreas Nickel

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Introduction

Iwasawa theory studies objects of arithmetic interest along infinite towers of fields. Let F be a number field and let p be a prime. Iwasawa studied Galois extensions F_∞ of F with Galois group $\Gamma = \mathrm{Gal}(F_\infty/F)$ isomorphic to $\mathbb{Z}_p$. Each closed subgroup of Γ is of the form Γ^{p^n} for some $n \in \mathbb{N}$, so that we obtain a tower of fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_\infty,$$

where F_n is a cyclic extension of F of degree p^n. Let A_n be the p-part of the class group of F_n. Iwasawa showed that for sufficiently large n the cardinality of A_n is given by

$$|A_n| = p^{\mu p^n + \lambda n + \nu}$$

for certain constants μ, λ and ν. These constants are invariants of the Iwasawa module $X = \mathrm{Gal}(M/F_\infty)$, where M is the maximal abelian unramified pro-p-extension of F_∞. Here 'Iwasawa module' means that it is a compact module over the completed group ring (the Iwasawa algebra) $\Lambda := \mathbb{Z}_p[\![\Gamma]\!]$. In fact, X is finitely generated and torsion as a Λ-module. This implies that there is a 'pseudo-isomorphism' (which in this situation just means that it has finite kernel and cokernel)

$$X \to \bigoplus_{i=1}^{s} \Lambda/p^{m_i} \oplus \bigoplus_{j=1}^{t} \Lambda/F_j^{n_j},$$

where the F_j are irreducible Weierstraß polynomials (note that Λ is isomorphic to a power series ring $\mathbb{Z}_p[\![T]\!]$). The characteristic ideal of X is the Λ-ideal generated by the characterisitic polynomial

$$p^\mu \cdot \prod_{j=1}^{t} F_j^{n_j},$$

where $\mu := \sum_{i=1}^{s} m_i$. Moreover, the λ-invariant of X is defined to be the degree of the characteristic polynomial, i.e. $\lambda = \sum_{j=1}^{t} n_j \deg(F_j)$. In particular, the invariants

μ and λ of X can be retrieved from the characteristic ideal. The latter might be interpreted as the 'first Chern class' of X. If the latter vanishes (and thus $\mu = \lambda = 0$), then X is finite, the invariant ν is its 'second Chern class' and is determined by $p^\nu = |X|$.

Let R be a local commutative Noetherian ring. In general, the 'm-th Chern class' of an (appropriate) R-module X is an element in $Z^m(R)$, the free abelian group generated by the prime ideals of R of height m. Note that the prime ideals of height 1 in Λ are precisely the ideals generated by p and the irreducible Weierstraß polynomials, whereas the maximal ideal of Λ is the unique prime ideal of height 2.

In classical terms, a 'main conjecture' asserts that two ideals in a formal power series ring agree. One ideal is the characteristic ideal of a natural torsion Iwasawa module, the other ideal is generated by a p-adic L-series. This might be seen as a statement about first Chern classes. However, a conjecture of Greenberg [5] asserts that many interesting Iwasawa modules are 'pseudo-null', i.e. their first Chern classes vanish. This is the main motivation to study higher Chern classes.

Suppose that R is a regular integral domain with field of fractions Q. The divisor homomorphism

$$\nu_1 : Q^\times = K_1(Q) \to Z^1(R)$$

in the case $m = 1$ is replaced by successively composing tame symbol maps in higher K-theory. This yields a map

$$\nu_m : \bigoplus K_m(Q) \to Z^m(R), \tag{1}$$

where the sum is over a certain set of 'Parshin chains', i.e. ordered sequences $(\eta_0, \ldots, \eta_{m-1})$ of points in $\mathrm{Spec}(R)$, with each η_i of height i and $\eta_i \in \overline{\{\eta_{i-1}\}}$. In the seminar, we are mainly interested in the case $m = 2$, where all elements in $K_2(Q)$ can be described in terms of Steinberg symbols by a theorem of Matsumoto. The map ν_2 then has a rather explicit description.

In [2] the authors are mainly interested in the following situation. Let p be an odd prime that splits into two primes $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ in an imaginary quadratic field E. Let $\tilde{E}$ be the compositum of all $\mathbb{Z}_p$-extensions of E and set $\Gamma := \mathrm{Gal}(\tilde{E}/E)$. Since Leopoldt's conjecture holds for E, we actually have $\Gamma \simeq \mathbb{Z}_p^2$. Let F be a finite abelian extension of E of order prime to p and such that F contains a primitive p-th root of unity. Set $K := \tilde{E}F$, $\Delta := \mathrm{Gal}(F/E)$ and $\mathcal{G} := \mathrm{Gal}(K/E) \simeq \Delta \times \Gamma$. Consider the Iwasawa module $X = \mathrm{Gal}(M/K)$, where M is the maximal abelian unramified pro-p-extension of K. In this situation one expects that X is pseudo-null as a $\Lambda := \mathbb{Z}_p[\![\Gamma]\!]$-module. Note that X is actually a $\mathbb{Z}_p[\![\mathcal{G}]\!] = \Lambda[\Delta]$-module so that we may decompose X into ψ-eigenspaces X^ψ, where ψ runs over the irreducible characters of Δ.

The Katz p-adic L-functions $\mathcal{L}_{\mathfrak{p},\psi}$ and $\mathcal{L}_{\overline{\mathfrak{p}},\psi}$ lie in the fraction field Q of $W[\![t_1, t_2]\!] \simeq W[\![\Gamma]\!]$, where W denotes the ring of Witt vectors of $\overline{\mathbb{F}}_p$. By Matsumoto's description of $K_2(Q)$ and the map ν_2 this is gives rise to an element c_2^{an} in $Z^2(W[\![\Gamma]\!])$. The main results of the paper [2] now states that $c_2^{\mathring{a}n}$ is equal to the sum of the second Chern classes of X^ψ and a certain dual of this module (after extending scalars to W):

$$c_2^{\mathring{a}n} = c_2(X_W^\psi) + c_2((X_W^{\omega\psi^{-1}})^\iota(1)).$$

A main ingredient of the proof is the two-variable main conjecture of Rubin [9], which states that the Katz p-adic L-functions are generators of certain $\mathfrak{p}$-ramified (resp. $\bar{\mathfrak{p}}$-ramified) Iwasawa modules.

Talks

1. **Introduction** **April 30**
   Give a survey on the content of the seminar and the aims of Iwasawa theory in general.

2. K_2 **of a ring** **May 7**
   Define $K_2(R)$ of a ring R as the kernel of the natural map $St(R) \to \mathrm{GL}(R)$ from the Steinberg group to the infinite general linear group (see [6, §5] or [3, §47A]). Show that $K_2(R)$ is precisely the center of the Steinberg group and thus abelian ([6, Theorem 5.1] or [3, Proposition 47.3]). Discuss the description in terms of universal central extensions [6, Theorem 5.10]. Introduce Steinberg symbols [6, §9; in particular Lemmas 9.7 and 9.8]. We need to know that $K_2(R)$ is generated by these symbols if R is a field [6, Corollary 9.13] or a commutative local ring [3, Theorem 47.39]. Some of the proofs rely on tedious computations, which you may wish to skip.

3. **Matsumoto's theorem and higher Chern classes** **May 14**
   State Matsumoto's theorem [6, Theorem 11.1] on K_2 of a field and explain, how the description in terms of universal central extensions is used in its proof (you can find a full proof in [6, §12]; what you should sketch is on p. 121). Present the content of [2, §2]. More precisely, define the m-th Chern class of a finitely generated R-module which is supported in codimension $\geq m$. Define the map ν_m above (1) with a particular focus on the cases $m = 1$ and $m = 2$ (the case $m \geq 3$ will actually play no role in our seminar). Show how $Z^2(R)$ can be described in terms of K_2-groups if R is a regular local ring such that Gersten's conjecture holds for R [2, (2.8)]. Prove [2, Proposition 2.5.1].

4. **Generalities on Iwasawa theory** **May 28**
   Recall the notions of 'reflexive module', 'pseudo-null' and 'pseudo-isomorphism' [8, Chapter V, §1]. Prove [8, Lemma 5.1.2] and deduce [8, Corollary 5.1.3]. Prove/Recall [8, Proposition 5.1.7(ii)] and rephrase the result in terms of first Chern classes. Recall the isomorphism $\mathcal{O}[\![\Gamma]\!] \simeq \mathcal{O}[\![T]\!]$ from [8, Proposition 5.3.5] (here $\Gamma \simeq \mathbb{Z}_p$) and specialize [8, Proposition 5.1.7(ii)] to the case of the Iwasawa algebra $\mathbb{Z}_p[\![\Gamma]\!]$ by using [8, Lemma 5.3.7]. Discuss the relation between Leopoldt's conjecture and the number of independent $\mathbb{Z}_p$-extensions of a number field [8, Proposition 10.3.20 (ii)] and prove [1, Lemma 3.1] (this is a more specialized result that we need later, but the method of proof is the same). Explain the statement and the state of the art of Greenberg's conjecture [2, §3.4].

5. **Duals of Iwasawa modules** **June 4**
   Present the content of the Appendix [2, Appendix A], which concerns Ext-groups and Iwasawa adjoints of modules over completed group rings. You may skip [2, Proposition A.4(b)] as it is not used later on. You may also assume that $\mathcal{G}$ is abelian.

6. **Spectral sequences and the core diagram** **June 18**
   The main goal of this talk is to construct the diagram in [2, Theorem 4.1.14]. The exposition in the follow-up paper [1] is much cleaner and even more general. So follow [1, §2] up to Proposition 2.11 (skip the claim '$D_P = I_P$', which will be proved in the next talk). Explain how to deduce [2, Theorem 4.1.14] from this by specializing $\Sigma = S_f$ (similarly, [2, Corollary 4.1.6] is a special case of [1, Proposition 2.7]).

7. **Some consequences of Leopoldt's conjecture** **June 25**
   Follow [2] from Proposition 4.1.15 to the end of §4. This includes some consequences of the (weak) Leopoldt conjecture and some lemmas that we need later on. For the Euler characteristic formulas (which appear in the proof of Lemma 4.3.1) you may refer to [7, 4.6.10 and 5.3.6] (see also [8, §7.3 and §8.7]).

8. **Reflection-type theorems for Iwasawa modules** **July 2**
   Follow [2, §5] up to Proposition 5.2.4. The short exact sequence of Theorem 5.2.1 is fundamental for the main result of the paper. §5.1 is interesting, though strictly speaking not necessary for the final talk.

9. **Katz p-adic L-functions and proof of the main result** **July 9**
   Deduce the main result (Theorem 5.2.5) and its Corollary 5.2.7. If you use the existence of Katz p-adic L-functions and Rubin's main conjecture [9] as black-boxes, then this is now rather easy (it mainly remains to prove Lemma 3.3.2 in §3.3). So it would be great to either sketch the construction of the relevant p-adic L-functions (as in [4, Chapter II, §4], for instance) or to give a sketch of Rubin's proof.

References

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