Matsumoto's Theorem and Higher Chern Classes

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Content

1. Recall from last week
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3. Higher Chern Classes

First we recall some notions and results from last week.
Let R be a commutative ring with 1. Then we have seen

$$K_0(R) = \mathrm{Groth}((\mathrm{Proj}_R^{\mathrm{fg}}, \oplus)/\sim)$$

$$K_1(R) = \mathrm{GL}(R)/E(R)$$

Here $\mathrm{GL}(R) = \varinjlim_n \mathrm{GL}_n(R)$ and $E(R)$ is the subgroup generated by the elementary matrices $E_{i,j}(r) = 1 + r \cdot e_{i,j}$ with $r \in R$ and $i \neq j$.

$$K_2(R) = \mathrm{Ker}(\mathrm{St}(R) \to E(R))$$

Here $\mathrm{St}(R)$ is the abstract group with generators $x_{i,j}(r)$ satisfying certain relations similar to the above elementary matrices.

In the case $R = F$ is a field, some simplifications occur:

$$K_0(F) = \mathbb{Z}$$

given by the F-dimension of the F-vector spaces.
The subgroup $E(F) \subset \mathrm{GL}(F)$ is exactly the subgroup $\mathrm{SL}(F)$ of matrices of determinant equal to 1.

$$K_1(F) = \mathrm{GL}(F)/\mathrm{SL}(F) \simeq F^\times.$$

At last, one has that

$$1 \to K_2(F) \to \mathrm{St}(F) \to \mathrm{SL}(F) \to 1$$

is a central universal extension.
This means that for any other central extension $1 \to N \to G \to \mathrm{SL}(F) \to 1$, there is a unique map $\mathrm{St}(F) \to G$ such that the right square in the following diagram commutes

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(F) & \longrightarrow & \mathrm{St}(F) & \longrightarrow & \mathrm{SL}(F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & \mathrm{SL}(F) & \longrightarrow & 1 \end{array}$$

This also induces a map $K_2(F) \to N$.
Moreover, we have also seen that the group $K_2(F)$ is generated by the symbols $\{r, s\}$ with $r, s \in F^\times$ modulo certain relations (this is also true for R a local ring).
Among these relations, there are bi-multiplicativity, skew-symmetry and $\{r, 1-r\} = 1 = \{r, -r\}$.

A result of Matsumoto determines this relation. Let F be a field. Write

$$A := \oplus_{r,s \in F^\times} e(r,s)^{\mathbb{Z}} / \sim,$$

where the relation $\sim$ is given by

$$\sim = \begin{cases} e(r_1 r_2, s) = e(r_1, s) e(r_2, s), \\ e(r, s_1 s_2) = e(r, s_1) e(r, s_2), \\ e(r, 1-r) = 1, \ \forall r \neq 0, 1 \end{cases}$$

Theorem (Matsumoto,69')

There is an isomorphism of abelian groups

$$\eta \colon A \to K_2(F), \quad e(r,s) \mapsto \{r,s\}.$$

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Remark

1. *Recall that the symbols $\{r,s\}$ also satisfy these relations, thus the map η is well-defined.*
2. *$e(r,s)$ is also skew-symmetric and $e(r,-r)=1$.*
   For example, note that

$$-r = \frac{1-r}{1-1/r} \in F^\times$$

$$\Rightarrow e(r,-r) = \frac{e(r,1-r)}{e(r,1-1/r)} = e(1/r, 1-1/r) = 1$$

Exercise for the skew-symmetry.

Sketch of proof of Matsumoto's Theorem: We write $D_n(F) \subset \mathrm{SL}_n(F)$ for the subgroup of diagonal matrices and $D(F) = \varinjlim_{n} D_n(F)$.

On one hand, recall the symbols $\{r, r'\} \in K_2(F)$ are defined as follows:

$$1 \to K_2(F) \to \mathrm{St}(F) \to \mathrm{SL}(F) \to 1,$$

take $d = \mathrm{diag}(r, \frac{1}{r}, 1)$ and $d' = \mathrm{diag}(r', 1, \frac{1}{r'})$ in $D(F)$ and take any liftings $\widehat{d}, \widehat{d'} \in \mathrm{St}(F)$ of d, d'.

Then $\{r, r'\} = \widehat{d}\widehat{d'}\widehat{d}^{-1}\widehat{d'}^{-1}$.

On the other hand, one can construct a central extension (recall $A = \oplus_{r,s \in F^\times} e(r,s)^{\mathbb{Z}} / \sim$)

$$1 \to A \to H \to D \to 1$$

as follows: write $H = D \times A$ whose group law is given by

$$\begin{aligned}(d, a)(d', a') &= (\mathrm{diag}(d_1, \cdots, d_n), a)(\mathrm{diag}(d'_1, \cdots, d'_n), a') \\ &:= (dd', aa' \prod_{i \geq j} e(d_i, d'_j)).\end{aligned}$$

One can then show

Lemma

For any $h = (d, a), h' = (d', a') \in H$, *one has*

$$hh'h^{-1}(h')^{-1} = (1, \prod_i e(d_i, d'_i)) \in A$$

Proof.

Direct computation: one has

$$h^{-1} = (d^{-1}, a^{-1} \prod_{i \geq j} e(d_j^{-1}, d_i)).$$

Thus one has

$$\begin{aligned} hh'h^{-1}(h')^{-1} &= (hh')(h'h)^{-1} \\ &= (dd', aa' \prod_{i \geq j} e(d_i, d'_j))(d'd, a'a \prod_{i \geq j} e(d'_i, d_j))^{-1} \end{aligned}$$

Proof.

$$hh'h^{-1}(h')^{-1} = (hh')(h'h)^{-1}$$

$$= \left(dd', aa' \prod_{i \geq j} e(d_i, d'_j)\right) \left(d'd, a'a \prod_{i \geq j} e(d'_i, d_j)\right)^{-1}$$

$$= \left(dd', aa' \prod_{i \geq j} e(d_i, d'_j)\right) \left((d'd)^{-1}, (a'a)^{-1} \prod_{i \geq j} e(d_j, d'_i) e((d'd)_j^{-1}, (d'd)_i)\right)$$

$$= (1, \prod_{i \geq j} e(d_i, d'_j) e(d_j, d'_i) \cdot \prod_{i \geq j} e((d'd)_j^{-1}, (d'd)_i) e((dd')_i, (d'd)_j^{-1}))$$

In the first product, $\prod_{i \geq j} e(d_i, d'_j) e(d_j, d'_i)$, for each fixed i, d_i appears in $e(d_i, d'_i), e(d_i, d'_{i-1}), \cdots, e(d_i, d'_1)$ as well as $e(d_i, d'_i), e(d_i, d'_{i+1}), \cdots, e(d_i, d'_n)$. Since $\prod_j d'_j = 1$, one sees that the product of these terms is equal to $e(d_i, d'_i)$.

Similarly, one can show that the second product is equal to 1. □

In particular, take $r, r' \in F^\times$ and $d = \mathrm{diag}(r, \frac{1}{r}, 1), d' = \mathrm{diag}(r', 1, \frac{1}{r'}) \in D(F)$ and $h, h' \in H$ any liftings of d, d'. Then one has

$$A \ni e(r, r') = e(r, r') \cdot e(\frac{1}{r}, 1) \cdot e(1, \frac{1}{r'}) = hh'h^{-1}(h')^{-1}$$

One can extend this central extension from $D(F)$ to the whole $\mathrm{SL}(F)$,

$$1 \to A \to G \to \mathrm{SL}(F) \to 1$$

and thus $H \subset G$, the inverse image of $D(F)$ under the map $G \to \mathrm{SL}(F)$ (whose construction is the most technical part of the proof, omitted here)

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and thus $H \subset G$, the inverse image of $D(F)$ under the map $G \to \mathrm{SL}(F)$ (whose construction is the most technical part of the proof, omitted here) From this we get the following commutative diagram

$$\begin{array}{ccccc} K_2(F) & \hookrightarrow & \mathrm{St}(F) & \twoheadrightarrow & \mathrm{SL}(F) \\ \downarrow \xi & & \downarrow \xi & & \downarrow \mathrm{Id} \\ A & \hookrightarrow & G & \twoheadrightarrow & \mathrm{SL}(F) \end{array}$$

now consider elements and liftings in these groups (recall $d = \mathrm{diag}(r, \frac{1}{r}, 1)$ and $d' = \mathrm{diag}(r', 1, \frac{1}{r'})$ and we require h, h' be images of $\widehat{d}, \widehat{d'}$)

$$\begin{array}{ccccc} (\{r, r'\} \in) K_2(F) & \hookrightarrow & (\widehat{d}, \widehat{d'} \in) \mathrm{St}(F) & \twoheadrightarrow & (d, d' \in) \mathrm{SL}(F) \\ \downarrow \xi & & \downarrow \xi & & \downarrow \mathrm{Id} \\ A & \hookrightarrow & (h, h' \in) G & \twoheadrightarrow & (d, d' \in) \mathrm{SL}(F) \end{array}$$

$$\begin{array}{ccccc}
(\{r,r'\}\in)K_2(F) & \hookrightarrow & (\widehat{d},\widehat{d'}\in)\mathrm{St}(F) & \twoheadrightarrow & (d,d'\in)\mathrm{SL}(F) \\
\downarrow \xi & & \downarrow \xi & & \downarrow \mathrm{Id} \\
A & \hookrightarrow & (h,h'\in)G & \twoheadrightarrow & (d,d'\in)\mathrm{SL}(F)
\end{array}$$

Let's look at the image

$$\xi(\{r,r'\}) = \xi(\widehat{d}\widehat{d'}\widehat{d}^{-1}\widehat{d'}^{-1}) = hh'h^{-1}(h')^{-1} = e(r,r') \in A.$$

This shows that ξ is inverse to the natural map $\eta\colon A \to K_2(F)$, $e(r,r') \mapsto \{r,r'\}$, thus it is an isomorphism of abelian groups. This is a sketch of proof of Matsumoto's theorem.

Theorem (Matsumoto,69')

There is an isomorphism of abelian groups

$$\eta\colon A = \oplus_{r,s\in F^\times} e(r,s)^{\mathbb{Z}}/\sim \to K_2(F), \quad e(r,s)\mapsto \{r,s\}.$$

This theorem can be reformulated as follows

Corollary

For any (multiplicative) abelian group A' and any map $e'\colon F^\times \times F^\times \to A'$ which is bi-multiplicative and $e'(r,1-r)=1$ for $r\neq 0,1$, there is a unique group morphism $K_2(F)\to A'$ taking $\{r,s\}$ to $e'(r,s)$.

Definition

Any map as above e' is called a Steinberg symbol on F.

The relation mentioned in the theorem reminds us perhaps of Hilbert symbols. Say $F = \mathbb{Q}_p$ with p a rational prime. Then the Hilbert symbol is given by

$$(\cdot,\cdot)_{\mathbb{Q}_p} : \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \to \{\pm 1\}$$

where $(r,s)_{\mathbb{Q}_p} = 1$ if the equation $rX^2 + sY^2 = 1$ has a solution in $\mathbb{Q}_p$ and $= -1$ otherwise.

One can work this out explicitly as follows: write $r = p^\alpha u$ and $s = p^\beta v$ with $u, v \in \mathbb{Z}_p^\times$, $\alpha, \beta \in \mathbb{Z}$, then

$$(r,s)_{\mathbb{Q}_p} = (p^\alpha u, p^\beta v)_{\mathbb{Q}_p} = \begin{cases} \left(\frac{(-1)^{\alpha\beta} u^\beta v^\alpha}{p}\right), & \text{if } p \neq 2 \\ (-1)^{\frac{u-1}{2}\cdot\frac{v-1}{2} + \alpha\frac{u^2-1}{8} + \beta\frac{v^2-1}{8} \pmod 2}, & \text{if } p = 2. \end{cases}$$

Here $(\frac{\cdot}{p})$ is the Legendre symbol: $(\frac{d}{p}) = 1$ if $d \in \mathbb{Z}$ prime to p is a square in $\mathbb{Z}/p$ is a square and $= -1$ otherwise.

Then one can show that Quadratic Reciprocity is equivalent to the identity

$$\prod_p (a,b)_{\mathbb{Q}_p} = 1, \quad \forall a, b \in \mathbb{Q}^\times.$$

More generally, if R is a DVR with $2 \in R^\times$, valuation $v\colon R^\times \to \mathbb{Z}$, fraction field $Q = Q(R)$ and residue field $k = R/\mathfrak{m}$.

Lemma

The map

$$d_v\colon Q^\times \times Q^\times \to k^\times, \quad (r,s) \mapsto (-1)^{v(r)v(s)} \frac{r^{v(s)}}{s^{v(r)}} \pmod{\mathfrak{m}}$$

is a Steinberg symbol.

Proof.

It is easy to see that this map is well-defined (the product is an element in $R^\times$) and the bi-multiplicativity is also clear. The identity $d_v(1-r,r) = 1$ is proved according to $v(r)$. For example, if $v(r) > 0$, then $r \in \mathfrak{m}$ and thus $v(1-r) = 0$. Therefore one has

$$d_v(1-r,r) = (-1)^{v(r)0}(1-r)^{v(r)}/r^0 \equiv 1 \pmod{\mathfrak{m}}.$$

The remaining cases are proved similarly. □

Remark

d_v is called the tame symbol of R. This induces a surjective map

$$\partial_2 \colon K_2(Q) \to K_1(k) = k^\times, \quad \{r, s\} \mapsto d_v(r, s).$$

This can be seen as part of the long exact sequence in algebraic K-theory

$$\cdots \to K_{i+1}(R) \to K_{i+1}(Q) \xrightarrow{\partial_{i+1}} K_i(k) \to K_i(R) \to K_i(Q) \to \cdots$$

We can write down the connecting map ∂_1 explicitly as follows

$$\partial_1 : K_1(Q) = Q^\times \to K_0(k) = \mathbb{Z}, \quad r \mapsto v(r).$$

Next we want to generalize these two maps to other rings than DVR.

Now let R be a regular local Noetherian integral domain, with maximal ideal $\mathfrak{m}$, fraction field $Q = Q(R)$. We then put

$$Y = \mathrm{Spec}(R), \quad Y^{(m)} = \{\eta \in Y | \eta \text{ of codim } m\},$$

$$Z^m(Y) = \oplus_{\eta \in Y^{(m)}} \mathbb{Z} \cdot \eta.$$

For a bounded complex $\mathcal{D}^\bullet$ of finitely generated R-modules exact in codimension $< m$ (that is, for any $\eta \in Y^{(<m)}$, the localization $\mathcal{D}^\bullet_\eta = \mathcal{D}^\bullet \otimes_R R_\eta$ is exact), we put

$$c_m(\mathcal{D}^\bullet)_\eta = \sum_i (-1)^i \mathrm{length}_{R_\eta} H^i(\mathcal{D}^\bullet_\eta), \quad \eta \in Y^{(m)},$$

$$c_m(\mathcal{D}^\bullet) = \sum_{\eta \in Y^{(m)}} c_m(\mathcal{D}^\bullet)_\eta \cdot \eta \in Z^m(Y).$$

$$c_m(\mathcal{D}^\bullet)_\eta = \sum_i (-1)^i \mathrm{length}_{R_\eta} H^i(\mathcal{D}^\bullet_\eta), \quad c_m(\mathcal{D}^\bullet) = \sum_{\eta \in Y^{(m)}} c_m(\mathcal{D}^\bullet)_\eta \cdot \eta \in Z^m(Y$$

Remark

1. *For a finitely generated R-module M, we view it as a complex concentrated at degree 0.*
2. *These sums are indeed finite sums: for any f.g. R-module M, the set of ideals $\{I \subset R | Im = 0 \text{ for some } m \in M\}$ has a maximal element $\mathfrak{p}$ under inclusion,*
   which we can show to be a prime ideal, thus we have an inclusion of R-modules $R/\mathfrak{p} \hookrightarrow M$.
   This shows that M has a finite filtration

   $$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

   such that $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$. Moreover, the support of $R/\mathfrak{p}$ gives $(R/\mathfrak{p})_\mathfrak{q} \neq 0$ iff $\mathfrak{p} \subset \mathfrak{q}$.

For two points $\eta_i \in Y^{(i)}$ and $\eta_{i-1} \in Y^{(i-1)}$ such that $\eta_i \in \overline{\{\eta_{i-1}\}}$ (that is, $\eta_i \supset \eta_{i-1}$ as ideals of R), the quotient R/η_i has fraction field, denoted by $k(\eta_i)$ and the localization $R_{\eta_{i-1},\eta_i} = (R/\eta_{i-1})_{\eta_i}$ is a 1-dimensional regular local integral domain (a DVR), with fraction field $k(\eta_{i-1})$, residue field $k(\eta_i)$ and valuation denoted by $v_{\eta_{i-1},\eta_i}\colon Q(R_{\eta_{i-1},\eta_i})^\times \to \mathbb{Z}$.
We next define two maps connecting the K-groups of these fields.

Definition

$$\nu_1 : \bigoplus_{\eta_0 \in Y^{(0)}} K_1(Q) = K_1(Q) = Q^\times \to Z^1(Y) = \bigoplus_{\eta_1 \in Y^{(1)}} \mathbb{Z} \cdot \eta_1$$

$$f \mapsto \mathrm{div}(f).$$

(height one prime ideals are principal ideals in UFDs) This induces an isomorphism

$$Q^\times / R^\times \xrightarrow{\sim} Z^1(Y).$$

For any $\eta_1 \in Y^{(1)}$, the localization R_{η_1} is a DVR, of fraction field $Q(R_{\eta_1}) = Q(R)$ and of residue field $k(\eta_1) = Q(R/\eta_1)$. We write the connecting map of algebraic K-groups

$$\partial_{2,\eta_1} = \partial_2 \colon K_2(Q) = K_2(Q(R_{\eta_1})) \to K_1(k(\eta_1)) = k(\eta_1)^\times.$$

For another $\eta_2 \in Y^{(2)}$ with $\eta_2 \supset \eta_1$, recall we have the valuation

$$v_{\eta_1,\eta_2} \colon Q((R/\eta_1)_{\eta_2})^\times = k(\eta_1)^\times \to \mathbb{Z}$$

Definition

$$\nu_2 \colon \bigoplus_{\eta_1 \in Y^{(1)}} K_2(Q) \to Z^2(Y), \quad a = (a_{\eta_1}) \mapsto \sum_{\eta_1 \in Y^{(1)}} \mathrm{div}_{\eta_1}(\partial_{2,\eta_1}(a_{\eta_1})).$$

where for an element $f \in k(\eta_1)^\times$, $\mathrm{div}_{\eta_1}(f) = \sum_{\eta_2 \in Y^{(2)} \cap \overline{\{\eta_1\}}} v_{\eta_1,\eta_2}(f) \cdot \eta_2$.

Definition

Let $m = 1, 2$ and $\mathcal{D}^\bullet$ be a bounded complex of f.g. R-modules, exact on co-dimension $m-1$. An element $a = (a_\eta) \in \oplus_{\eta \in Y^{(m-1)}} K_m(Q)$ is an m-th characteristic symbol for $\mathcal{D}^\bullet$ if

$$\nu_m(a) = c_m(\mathcal{D}^\bullet).$$

For example for $m = 1$, an element $f \in K_1(Q) = Q^\times$ is a 1st char symbol for $\mathcal{D}^\bullet$ if $\mathrm{div}(f) = c_1(\mathcal{D}^\bullet) = \sum_{\eta_1 \in Y^{(1)}} \eta_1 \cdot \sum_i (-1)^i \mathrm{length}_{R_{\eta_1}}(H^i(\mathcal{D}^\bullet_{\eta_1}))$
This is a generalization of the notion of characteristic power series $p^\nu \prod_j F_j(T)^{e_j}$ for a torsion f.g. Iwasawa module M over $\mathbb{Z}_p[[T]]$ (we will say more about this next time).

In the case $m = 2$, we have the following

Proposition

Let $f_1, f_2 \in R$ be two prime elements such that $f_1/f_2 \notin R^\times$. Then a 2nd char symbol for $R/(f_1, f_2)$ is given by $a = (a_{\eta_1})$ with

$$a_{\eta_1} = \begin{cases} \{f_2, f_1\}, & \text{if } \eta_1 = (f_1); \\ 1, & \text{otherwise.} \end{cases}$$

Proof.

This follows basically from the definitions. Note that the ideal (f_1, f_2) is a prime ideal of height 2, therefore

$$\begin{aligned} c_2(R/(f_1, f_2)) &= \sum_{\eta_2 \in Y^{(2)}} \mathrm{length}_{R_{\eta_2}}((R/(f_1, f_2))_{\eta_2}) \cdot \eta_2 \\ &= \sum_{(f_1, f_2) \subset \eta_2 \in Y^{(2)}} \mathrm{length}_{R_{\eta_2}}((R/(f_1, f_2))_{\eta_2}) \cdot \eta_2 \\ &= 1 \cdot (f_1, f_2) \in Z^2(Y). \end{aligned}$$

Proof.

$$v_2(a) = \sum_{\eta_1 \in Y^{(1)}} \mathrm{div}_{\eta_1}(\partial_{2,\eta_1}(a_{\eta_1}))$$
$$= \sum_{\eta_1 \in Y^{(1)}} \sum_{\eta_1 \subset \eta_2 \in Y^{(2)}} v_{\eta_1,\eta_2}(\partial_{2,\eta_1}(a_{\eta_1})) \cdot \eta_2$$
$$\overset{\eta_1=(f_1)}{=} \sum_{\eta_1 \subset \eta_2 \in Y^{(2)}} v_{\eta_1,\eta_2}(\partial_{2,\eta_1}(\{f_2, f_1\})) \cdot \eta_2$$
$$= \sum_{\eta_1 \subset \eta_2 \in Y^{(2)}} v_{\eta_1,\eta_2}\left((-1)^{v_{\eta_1}(f_2)v_{\eta_1}(f_1)} f_2^{v_{\eta_1}(f_1)} / f_1^{v_{\eta_1}(f_2)}\right) \cdot \eta_2$$

Here $v_{\eta_1} = v_{\eta_0,\eta_1} : Q(R_{\eta_0,\eta_1})^\times \to \mathbb{Z}$ is the valuation determined by $\eta_1 = (f_1)$ ($\eta_0 = (0)$). Therefore $v_{\eta_1}(f_1) = 1$ and $v_{\eta_1}(f_2) = 0$. □

Proof.

So we have

$$\nu_2(a) = \sum_{\eta_1 \subset \eta_2 \in Y^{(2)}} v_{\eta_1,\eta_2}\left((-1)^{v_{\eta_1}(f_2)v_{\eta_1}(f_1)} f_2^{v_{\eta_1}(f_1)} / f_1^{v_{\eta_1}(f_2)}\right) \cdot \eta_2$$

$$= \sum_{\eta_1 \subset \eta_2 \in Y^{(2)}} v_{\eta_1,\eta_2}\left((-1)^{1\cdot 0} f_2^1 / f_1^0\right) \cdot \eta_2 = \sum_{\eta_1 \subset \eta_2 \in Y^{(2)}} v_{\eta_1,\eta_2}(f_2) \cdot \eta_2$$

Again $v_{\eta_1,\eta_2}(f_2)$ is non-zero only for those $f_2 \in \eta_2$, that is, only for those $(f_1, f_2) \subset \eta_2$. But η_2 is of height 2, only one such η_2 exists, that is, $\eta_2 = (f_1, f_2)$, which gives

$$\nu_2(a) = 1 \cdot \eta_2 = 1 \cdot (f_1, f_2) = c_2(R/(f_1, f_2)).$$

□

The following element is also a 2nd char symbol for $R/(f_1, f_2)$, $a' = (a'_\eta)$ with $a'_\eta = \{f_1, f_2\}$ if $\eta = (f_2)$, and $= 1$ otherwise.

Next we talk about the surjectivity of the map $\nu_2 \colon \oplus_{\eta_1 \in Y^{(1)}} \to Z^2(Y)$. Gersten's conjecture says that the following sequence is exact

$$1 \to K_2(R) \to K_2(Q) \xrightarrow{\vartheta_2} \bigoplus_{\eta_1 \in Y^{(1)}} k(\eta_1)^\times \xrightarrow{\vartheta_1} \bigoplus_{\eta_2 \in Y^{(2)}} \mathbb{Z} \to 0.$$

Here the component of ϑ_2 at η_1 is the map

$$\partial_{2,\eta_1} \colon K_2(Q) \to K_1(k(\eta_1)) = k(\eta_1)^\times$$

and the component of ϑ_1 at (η_1, η_2) is the valuation

$$v_{\eta_1,\eta_2} \colon K_1(k(\eta_1)) = k(\eta_1)^\times \to K_0(k(\eta_2)) = \mathbb{Z}.$$

This conjecture is proved in many cases: Dennis-Stein for R a DVR, Quillen for R of finite type over a field, Gillet-Levine and Bloch for R of finite type and smooth over a mixed char. DVR, Reid-Sherman for $R = A[[T_1, \cdots, T_r]]$ with A a complete DVR, etc.

The two exact sequences ($\eta_1 \in Y^{(1)}$)

$$1 \to K_2(R_{\eta_1}) \to K_2(Q) \to k(\eta_1)^\times \to 1,$$

$$1 \to K_2(R) \to K_2(Q) \xrightarrow{\vartheta_2} \bigoplus_{\eta_1 \in Y^{(1)}} k(\eta_1)^\times \xrightarrow{\vartheta_1} \bigoplus_{\eta_2 \in Y^{(2)}} \mathbb{Z} \to 0$$

gives the isomorphism

$$\overline{\nu_2}\colon \frac{\prod'_{\eta_1 \in Y^{(1)}} K_2(Q)}{K_2(Q) \prod_{\eta_1 \in Y^{(1)}} K_2(R_{\eta_1})} \simeq Z^2(Y) = \bigoplus_{\eta_2 \in Y^{(2)}} \mathbb{Z} \cdot \eta_2,$$

where

$$\prod_{\eta_1 \in Y^{(1)}}' K_2(Q) = \left\{ (a_{\eta_1}) \in \prod_{\eta_1 \in Y^{(1)}} K_2(Q) | a_{\eta_1} \in K_2(R_{\eta_1}) \text{ for a.a. } \eta_1 \right\}$$

for the restricted product and $K_2(Q) \hookrightarrow \prod'_{\eta_1 \in Y^{(1)}} K_2(Q)$ the diagonal embedding (for any $\{f, g\} \in K_2(Q)$, $\{f, g\} \in K_2(R_{\eta_1})$ for almost all $\eta_1 \in Y^{(1)}$).

The map

$$\overline{\nu_2}\colon \frac{\prod'_{\eta_1\in Y^{(1)}} K_2(Q)}{K_2(Q)\prod_{\eta_1\in Y^{(1)}} K_2(R_{\eta_1})} \to Z^2(Y) = \bigoplus_{\eta_2\in Y^{(2)}} \mathbb{Z}\cdot\eta_2$$

in fact comes from $\nu_2\colon \bigoplus_{\eta_1\in Y^{(1)}} K_2(Q) \to Z^2(Y)$ because we have for $f, g \in R_{\eta_1}^{\times}$, $v_{\eta_1}(f) = v_{\eta_1}(g) = 0$ and thus

$$\begin{aligned} v_{\eta_1,\eta_2}(\partial_{2,\eta_1}(\{f,g\}))\cdot\eta_2 &= v_{\eta_1,\eta_2}((-1)^{v_{\eta_1}(f)v_{\eta_1}(g)}\frac{f^{v_{\eta_1}(g)}}{g^{v_{\eta_1}(f)}})\cdot\eta_2 \\ &= 0\cdot\eta_2 \end{aligned}$$

and similarly one can prove that the diagonal image of $K_2(Q)$ is taken to 0 by ν_2.

Thank you for your attention!