Some consequences of Leopoldt's conjecture (II)

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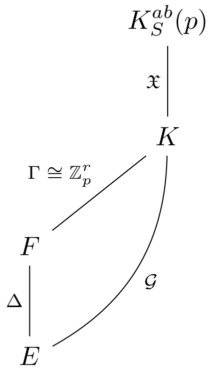
Outline

1: Set-up

- p: prime
- F/E: Galois extension of number fields
  $\Delta := \mathrm{Gal}(F/E)$
- K: Galois extension of both F and E such that $\Gamma := \mathrm{Gal}(K/F) \cong \mathbb{Z}_p^r$ for some $r \geq 1$
  $\mathcal{G} := \mathrm{Gal}(K/E)$.
- $\Lambda := \mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T_1, \ldots, T_r]]$
  $\Omega := \mathbb{Z}_p[[\mathcal{G}]]$
- S: a finite set of places of E, $S \supseteq S_p \cup S_\infty$
  K_S: the maximal extension of K unramified outside of S
  $\mathfrak{X} := G_{K,S}^{ab}(p) = H_1(G_{K,S}, \mathbb{Z}_p)$



Let $\mathfrak{p} \in S_f$

- $\mathcal{K}_\mathfrak{p} := \mathbb{Z}_p[[\mathcal{G}/\mathcal{G}_\mathfrak{p}]]$, an Ω-module (in general $\mathcal{G}_\mathfrak{p} \ntriangleleft \mathcal{G}$)
  $\mathcal{K} := \bigoplus_{\mathfrak{p} \in S_f} \mathcal{K}_\mathfrak{p}$
  $\mathcal{K}_0 := \ker(aug\colon \mathcal{K} \to \mathbb{Z}_p)$
- $\mathfrak{D}_\mathfrak{p} := (G_{K_\mathfrak{p}})^{ab}(p)$
  $\mathfrak{I}_\mathfrak{p} :=$ inertia subgroup of $\mathfrak{D}_\mathfrak{p}$
  $D_\mathfrak{p} := \Omega \hat{\otimes}_{\mathbb{Z}_p[[\mathcal{G}_\mathfrak{p}]]} \mathfrak{D}_\mathfrak{p}$
  $I_\mathfrak{p} := \Omega \hat{\otimes}_{\mathbb{Z}_p[[\mathcal{G}_\mathfrak{p}]]} \mathfrak{I}_\mathfrak{p}$

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For a locally compact Λ-(or Ω-)module, we abbreviated

$$E^i_\Lambda(M) := \mathrm{Ext}^i_\Lambda(M, \Lambda)$$

and set $M^* := E^0_\Lambda(M) = \mathrm{Hom}_\Lambda(M, \Lambda)$.

2: Useful lemmas

We saw:

Lemma (4.2.2)

Let $\mathfrak{p} \in S^f$ and suppose $\Gamma_\mathfrak{p} \neq 0$. Consider

- $\varepsilon_\mathfrak{p} = 0$ if $K_\mathfrak{p}$ contains the unram. $\mathbb{Z}_p$-extension of $E_\mathfrak{p}$, 1 otherwise.
- $\varepsilon'_\mathfrak{p} = \varepsilon_\mathfrak{p} \delta_{r_\mathfrak{p},1}$, where $r_\mathfrak{p} = \operatorname{rank} \mathbb{Z}_p$

If $\varepsilon'_\mathfrak{p} = 1$, assume that $K \supseteq \mu_{p^\infty}$. Then the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_\mathfrak{p} & \longrightarrow & D_\mathfrak{p} & \longrightarrow & \mathcal{K}_\mathfrak{p}^{\varepsilon_\mathfrak{p}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \iota & & \\ 0 & \longrightarrow & I_\mathfrak{p}^{**} & \longrightarrow & D_\mathfrak{p}^{**} & \longrightarrow & \mathcal{K}_\mathfrak{p}^{\varepsilon'_\mathfrak{p}} & \longrightarrow & 0 \end{array}$$

commutes and has exact rows. If $\varepsilon'_\mathfrak{p} = 1$, then $\iota = \mathrm{Id}$.

Relation $E_\Lambda^j(\mathcal{K}) \leftrightarrow E_\Lambda^j(\mathcal{K}_0)$?

Lemma (4.2.4)

a. For $1 \le j < r-1$, we have $E_\Lambda^j(\mathcal{K}) \cong E_\Lambda^j(\mathcal{K}_0)$
   For $j > r$, we have $E_\Lambda^j(\mathcal{K}) = E_\Lambda^j(\mathcal{K}_0) = 0$.

b. If $r \neq r_\mathfrak{p}$ for all $\mathfrak{p} \in S_f$, then
   - $E_\Lambda^r(\mathcal{K}) \cong E_\Lambda^r(\mathcal{K}_0) = 0$
   - The sequence $0 \to E_\Lambda^{r-1}(\mathcal{K}) \to E_\Lambda^{r-1}(\mathcal{K}_0) \to \mathbb{Z}_p \to 0$ is exact

c. If $r = r_\mathfrak{p}$ for some $\mathfrak{p} \in S_f$, then
   - $E_\Lambda^{r-1}(\mathcal{K}) \cong E_\Lambda^{r-1}(\mathcal{K}_0)$
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Lemma (4.2.4)

ⓐ For $1 \leq j < r-1$, we have $E^j_\Lambda(\mathcal{K}) \cong E^j_\Lambda(\mathcal{K}_0)$
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Idea:

1. Long Ext sequence of $0 \to \mathcal{K}_0 \to \mathcal{K} \to \mathbb{Z}_p \to 0$
2. $E^j_\Lambda(\mathbb{Z}_p) = \mathbb{Z}_p^{\delta_{r,j}}$ (A.13)
3. $E^j_\Lambda(\mathcal{K}) = \bigoplus_{\mathfrak{p} \in S_f} (\mathcal{K}^\iota_\mathfrak{p})^{\delta_{r_\mathfrak{p},j}}$ (4.1.13)

3: Eigenspaces

Simplification of the setting:

- Δ abelian
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We are interested in characters $\psi \colon \Delta \to \overline{\mathbb{Q}_p}^*$. Let

- $\mathcal{O}_\psi := \mathbb{Z}_p[\psi]$. We have a surjection $\mathbb{Z}_p[\Delta] \twoheadrightarrow \mathcal{O}_\psi$.
- $\Lambda_\psi := \mathcal{O}_\psi[[\Gamma]] = \Lambda[\psi]$
- For a $\mathbb{Z}_p[\Delta]$-module M, let $M^\psi = M \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\psi$

Note that $\Omega^\psi = \Lambda[\Delta] \otimes_{\mathbb{Z}_p[\Delta]} \mathcal{O}_\psi \cong \Lambda_\psi$ (as compact $\mathcal{O}_\psi$-algebras).

Remark: letting $\mathcal{Q} := \mathrm{Frac}(\Lambda)$, we have (Wedderburn)

$$\mathcal{Q}[\Delta] \cong \bigoplus_{\psi \in \mathrm{Irr}(\Delta)/\sim} \mathcal{Q}(\psi)$$

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$$\Omega = \Lambda[\Delta] \cong \bigoplus_{\psi \in \mathrm{Irr}(\Delta)/\sim} \Lambda[\psi],$$

hence Ω-modules really decompose as

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In general, however, we only know $\Omega \hookrightarrow \bigoplus_{\psi} \Lambda[\psi]$ (maximal order).

Lemma (4.3.1)

a. If weak Leopoldt holds for K, then $\operatorname{rank}_{\Lambda_\psi} \mathfrak{X}^\psi = r_2(E) + r_1^\psi(E)$, where $r_1^\psi(E) =$ no. of real places of E at which ψ is odd (i.e. $\psi|_{\Delta_\mathfrak{p}} \neq 1$).
b. Let $\mathfrak{p} \in S^f$. If $\Gamma_\mathfrak{p} \neq 0$ or $\psi|_{\Delta_\mathfrak{p}} \neq 1$, then $\operatorname{rank}_{\Lambda_\psi} D_\mathfrak{p}^\psi = [E_\mathfrak{p} : \mathbb{Q}_p]$.

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- (b) Let $\mathfrak{p} \in S^f$. If $\Gamma_\mathfrak{p} \neq 0$ or $\psi|_{\Delta_\mathfrak{p}} \neq 1$, then $\mathrm{rank}_{\Lambda_\psi} D_\mathfrak{p}^\psi = [E_\mathfrak{p} : \mathbb{Q}_p]$.

General proof: local and global class field theory. It uses Nekovář's Euler-Poincaré characteristic formulas:

$$\sum_{j=0}^{2}(-1)^{j-1} \mathrm{rank}_{\Lambda_\psi} H^j(G_{E,\Sigma}, B_\psi)^\vee = \sum_{v \in S_\infty} \mathrm{rank}_{\Lambda_\psi} (\Omega^\psi(1))^{G_{E_v}}$$

and

$$\sum_{j=0}^{2}(-1)^{j} \mathrm{rank}_{\Lambda_\psi} H^j(G_{E_\mathfrak{p}}, B_\psi)^\vee = [E_\mathfrak{p} : \mathbb{Q}_p].$$

(Very) simplified proof: $r = 1, F = E$

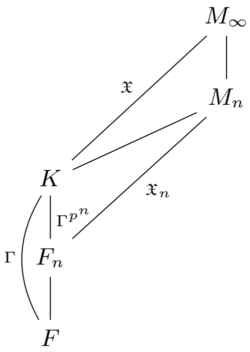
a) WeakLeopoldt(K) $\Rightarrow \operatorname{rank}_\Lambda \mathfrak{X} = r_2(F)$

Let:

- $F_n := K^{\Gamma^{p^n}}$
- $M_n :=$ maximal abelian pro-p extension of F_n unramified outside S. One has

  $$F_\infty \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\infty = (K_S)^{ab}(p).$$

- $\mathfrak{X}_n := \mathrm{Gal}(M_n/F_n)$, so $\mathfrak{X} = \varprojlim_n \mathfrak{X}_\mathfrak{n}$



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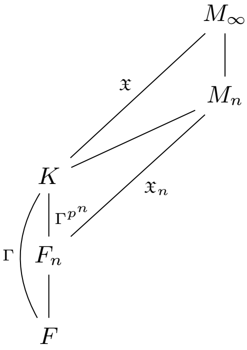
Then $\mathfrak{X}_{\Gamma^{p^n}} = \mathrm{Gal}(M_n/K)$ and therefore

$$0 \to \mathfrak{X}_{\Gamma^{p^n}} \to \mathfrak{X}_n \to \Gamma^{p^n} \to 0 \qquad (1)$$

is exact.



Recall that

$$r_2(F_n) + 1 + \mathfrak{d}_p(F_n) = \text{no. of indep. } \mathbb{Z}_p\text{-extensions of } F_n = \operatorname{rank}_{\mathbb{Z}_p} \mathfrak{X}_n$$

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- On the left, $r_2(F_n) = p^n r_2(F)$ ($\mathbb{Z}_p$-extensions are unramified at archimedean places)
- On the right, the sequence (1) $0 \to \mathfrak{X}_{\Gamma^{p^n}} \to \mathfrak{X}_n \to \Gamma^{p^n} \to 0$ shows that $\operatorname{rank}_{\mathbb{Z}_p} \mathfrak{X}_{\Gamma^{p^n}} = \operatorname{rank}_{\mathbb{Z}_p} \mathfrak{X}_n - 1$.

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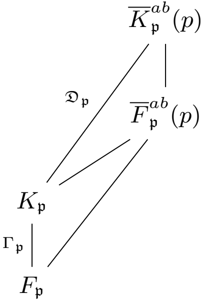
Therefore, $\operatorname{rank}_{\mathbb{Z}_p} \mathfrak{X}_{\Gamma^{p^n}} = p^n r_2(F) + \mathfrak{d}_p(F_n)$.

General fact: for all n large enough, $\operatorname{rank}_{\mathbb{Z}_p} \mathfrak{X}_{\Gamma^{p^n}} = p^n \operatorname{rank}_\Lambda \mathfrak{X} + c$ for some c independent of n. Hence $\operatorname{rank}_\Lambda \mathfrak{X} = r_2(F)$ by Weak Leopoldt.

b) $\Gamma_{\mathfrak{p}} \neq 0 \Rightarrow \operatorname{rank}_{\Lambda} D_{\mathfrak{p}} = [F_{\mathfrak{p}} : \mathbb{Q}_p]$
Let $\Lambda_{\mathfrak{p}} := \mathbb{Z}_p[[\Gamma_{\mathfrak{p}}]]$ (Iwasawa algebra). Note that

$$\operatorname{rank}_{\Lambda} D_{\mathfrak{p}} = \operatorname{rank}_{\Lambda} \Lambda \otimes_{\Lambda_{\mathfrak{p}}} \mathfrak{D}_{\mathfrak{p}} = \operatorname{rank}_{\Lambda_{\mathfrak{p}}} \mathfrak{D}_{\mathfrak{p}}.$$

Let $\overline{K}_{\mathfrak{p}}^{ab}(p)$ (resp. $\overline{F}_{\mathfrak{p}}^{ab}(p)$) be the maximal abelian pro-p extension of $K_{\mathfrak{p}}$ (resp. $F_{\mathfrak{p}}$). In particular, $\overline{F}_{\mathfrak{p}}^{ab}(p) \subseteq \overline{K}_{\mathfrak{p}}^{ab}(p)$.



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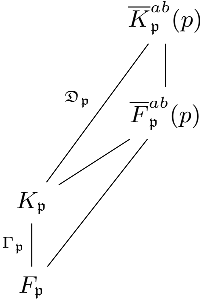
Similarly to before, $(\mathfrak{D}_{\mathfrak{p}})_{\Gamma_{\mathfrak{p}}} = \operatorname{Gal}(\overline{F}_{\mathfrak{p}}^{ab}(p)/K_{\mathfrak{p}})$, so

$$0 \to (\mathfrak{D}_{\mathfrak{p}})_{\Gamma_{\mathfrak{p}}} \to \operatorname{Gal}(\overline{F_{\mathfrak{p}}}^{ab}(p)/F_{\mathfrak{p}}) \to \Gamma_{\mathfrak{p}} \to 0$$

is exact.



Hence $\mathrm{rank}_{\mathbb{Z}_p}(\mathfrak{D}_\mathfrak{p})_{\Gamma_\mathfrak{p}} = \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Gal}(\overline{F_\mathfrak{p}}^{ab}(p)/F_\mathfrak{p}) - \mathrm{rank}_{\mathbb{Z}_p} \Gamma_\mathfrak{p}$

But $\mathrm{rank}_{\mathbb{Z}_p} \mathrm{Gal}(\overline{F_\mathfrak{p}}^{ab}(p)/F_\mathfrak{p}) =$ no. of indep. $\mathbb{Z}_p$-extensions of $F_\mathfrak{p}$
$= [F_\mathfrak{p} : \mathbb{Q}_p] + 1$ (local CFT)

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General fact:

$$\begin{aligned}\mathrm{rank}_{\Lambda_\mathfrak{p}}\mathfrak{D}_\mathfrak{p} &= \mathrm{rank}_{\mathbb{Z}_p}(\mathfrak{D}_\mathfrak{p})_{\Gamma_\mathfrak{p}} - \mathrm{rank}_{\mathbb{Z}_p}(\mathfrak{D}_\mathfrak{p})^{\Gamma_\mathfrak{p}}\\ &= [F_\mathfrak{p} : \mathbb{Q}_p] \qquad - 0\end{aligned}$$

∎

Thank you for your attention