Research seminar:
The Brumer–Stark Conjecture

(after Dasgupta and Kakde)

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Introduction

Let L/K be a finite Galois extension of number fields with Galois group G. To each finite set S of places of K containing all the archimedean places, one can associate a so-called 'Stickelberger element' θ_S in the centre of the group algebra $\mathbb{C}[G]$. This element is constructed from values at $s = 0$ of the S-truncated Artin L-series attached to the complex irreducible characters of G. By a result of Siegel [15] one knows that θ_S always has rational coefficients.

Let μ_L and cl_L be the roots of unity and the class group of L, respectively. Suppose that S also contains all places of K which ramify in L. Then it was independently shown by Pi. Cassou-Noguès [4], Deligne–Ribet [8] and Barsky [1] that for abelian G one has

$$\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_L)\theta_S \subseteq \mathbb{Z}[G],$$

where $\mathrm{Ann}_R(M)$ denotes the annihilator ideal of M regarded as a module over the ring R. In other words, the coefficients of θ_S are almost integral. Now Brumer's conjecture simply asserts that $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_L)\theta_S$ annihilates cl_L. In the case $K = \mathbb{Q}$ Brumer's conjecture is just Stickelberger's theorem from the late 19th century [17].

We henceforth assume that G is abelian. Harold Stark suggested the following refinement of Brumer's conjecture (discussed by Tate in [18]). Let w_L be the cardinality of μ_L and fix a fractional ideal $\mathfrak{a}$ in L. Then the Brumer–Stark conjecture not only predicts that $\mathfrak{a}^{w_L\theta_S}$ becomes principal, but also gives precise information about a generator of that ideal.

There is a variant of this conjecture as follows. Let T be a second finite set of places of K such that S and T are disjoint and such that there is no non-trivial root of unity in L that is congruent to 1 modulo all places w of L above a place v in T. Then

$$\delta_T := \prod_{v \in T}(1 - \mathrm{Frob}_w^{-1} N(v)) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu_L)$$

and the $\mathbb{Z}[G]$-ideal generated by these δ_T's indeed coincides with $\mathrm{Ann}_{\mathbb{Z}[G]}(\mu_L)$. In particular, one has that

$$\theta_S^T \in \mathbb{Z}[G].$$

Let cl_L^T be the ray class group of L with conductor equal to the product over all primes above T. Then (a variant of) the Brumer–Stark conjecture simply asserts that

$$\theta_S^T \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{cl}_L^T). \tag{1}$$

Away from 2, this conjecture is equivalent to the original Brumer–Stark conjecture. It is also clear that this conjecture for all admissible sets T implies Brumer's conjecture above.

It is also known that it suffices to consider CM-extensions L/K, i.e. K is totally real, L is totally complex and complex conjugation induces a unique automorphism $\sigma \in G$. So let us assume this. Fix an odd prime p and set $A_L^T := (\mathrm{cl}_L^T \otimes \mathbb{Z}_p)^-$, where for any $\mathbb{Z}_p[G]$-module M we denote by M^- the maximal submodule of M upon which σ acts by -1. Moreover, we denote its Pontryagin dual $\mathrm{Hom}(M, \mathbb{Q}_p/Z_p)$ by $M^\vee$ endowed with the contragradient G-action.

Dasgupta and Kakde [6] prove the following groundbreaking result.

Theorem 1 *One has* $(\theta_S^T)^\sharp \in \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}((A_L^T)^\vee)$.

Here $x \mapsto x^\sharp$ denotes the involution on $\mathbb{Z}_p[G]$ that maps each $g \in G$ to its inverse and Fitt denotes the (initial) Fitting ideal. Since $\mathrm{Fitt}(M) \subseteq \mathrm{Ann}(M)$ for any M and $\mathrm{Ann}(M^\vee) = \mathrm{Ann}(M)^\sharp$ for every finite $\mathbb{Z}_p[G]^-$-module M, this is actually a refinement of the p-primary part of (1).

The aim of this seminar is to understand the proof of Theorem 1. We briefly sketch some of the main steps.

In the following we often suppress the dependence of certain objects on S and T. In particular, we simply write θ for any Stickelberger element. Moreover, we warn the reader that not every statement is correct in a strict sense as we mainly want to fix ideas.

If a finitely generated $\mathbb{Z}_p[G]^-$-module M admits a 'quadratic presentation', i.e. a presentation of the form

$$(\mathbb{Z}_p[G]^-)^n \xrightarrow{H} (\mathbb{Z}_p[G]^-)^n \to M \to 0$$

for some n, then the Fitting ideal of M is principal and generated by $\det(H)$. If M is finite, then in addition one has

$$|M| = |\mathbb{Z}_p[G]^-/\mathrm{Fitt}(M)| \tag{2}$$

and

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(M^\vee) = \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(M)^\sharp. \tag{3}$$

Suppose that A_L^T admits a quadratic presentation (e.g. if $p \nmid |G|$) and that we can show an inclusion

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(A_L^T) \subseteq (\theta).$$

Then the analytic class number formula and (2) imply that this inclusion is indeed an equality. Then Theorem 1 follows by (3).

In general, however, none of A_L^T and $(A_L^T)^\vee$ has a quadratic presentation. To remedy this, Dasgupta and Kakde replace these modules by a Selmer module Sel(L/K) introduced by Burns, Kurihara and Sano [3] that projects onto $(A_L^T)^\vee$ and always admits a quadratic presentation. An elaborate refinement of the above argument then shows that the inclusions

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\mathrm{Sel}(L/K)) \subseteq (\theta) \tag{4}$$

for *all* CM-extensions L/K imply equality for all these extensions.

The Selmer group $\mathrm{Sel} = \mathrm{Sel}(L/K)$ is actually (almost) the transpose (in the sense of Jannsen [12]) of the module ∇ of Ritter and Weiss [13], where the authors constructed 'Tate sequences' for small S. This will become essential later in the proof. In particular, one has

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\mathrm{Sel}) = \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(\nabla)^\sharp.$$

In order to prove the inclusion (4) the authors work with group ring valued Hilbert modular forms (as considered by Wiles [19] and further developed very recently by Silliman [16]). Let $k \geq 1$ be an odd integer. If M_k denotes the space of Hilbert modular forms of weight k, then a group ring valued Hilbert modular form is an $f \in M_k \otimes \mathbb{Q}[G]^-$ with integral Fourier coefficients and such that for each odd character ψ of G the specialization $\psi(f)$ is a classical form of nebentypus ψ.

As in the classical setting one has Eisenstein series E_k whose constant terms at a cusp are now given in terms of Stickelberger elements. A main part of the proof is the construction of a cuspidal form F_k such that

$$F_k \equiv E_k \bmod \theta. \tag{5}$$

In fact, this is only true if L/K is unramified. In general, one has to replace E_k by a certain alternating sum of Eisenstein series. To construct F_k, the authors apply Ribet's method which roughly goes as follows:

If there is a modular form G of the same weight k and such that its constant term at the same cusp is 1, then one simply puts $F_k = E_k - \theta G$. However, it is in general only possible to construct such a G for sufficiently large k. For small k one then needs to construct a form V whose weight is the difference of the weights of G and E_k, whose constant term is 1, and such that $V \equiv 1 \bmod \theta$. Then

$$F_k := E_k V - \theta G.$$

Let $\mathbb{T}$ denote the (p-completion of the) Hecke algebra of the module of weight k cuspidal group ring valued modular forms. Since E_k is an eigenform, the congruence (5) implies that there is a homomorphism

$$\phi : \mathbb{T} \to \mathbb{Z}_p[G]^-/\theta$$

which maps a Hecke operator to the corresponding eigenvalue of E_k. Let I denote the kernel of ϕ (the 'Eisenstein ideal'). The theory of Wiles and Hida associates to ϕ a certain two-dimensional Galois representation of G_K. The latter gives rise to a faithful $\mathbb{T}$-module B and a cohomology class

$$\kappa \in H^1(G_K, B/IB)$$

which is unramified outside p and T. Assume for simplicity that κ is also unramified at p (in general one has to alter the map ϕ slightly). By class field theory this gives rise to a surjective homomorphism

$$A_L^T \to B/IB.$$

Since B is a faithful $\mathbb{T}$-module and $\mathbb{T}/I \simeq \mathbb{Z}_p[G]^-/\theta$, general properties of Fitting ideals now show that

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(A_L^T) \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(B/IB) \subseteq (\theta)$$

as desired. Actually, we are oversimplifying here as in general one has to work with the Ritter–Weiss module ∇ rather than A_L^T.

We now outline the talks in more detail. We start with a research talk by Samit Dasgupta. Since not all of us are familiar with Hilbert modular forms, talk 2 and 3 are devoted to some basics of such forms. Then we start reading [6]. Talks 3,5 and 6 are independent of the two previous talks. Hilbert modular forms again enter the stage in talk 7 and are extensively used also in talks 8 and 9. These three talks do not depend upon talks 5 and 6. Indeed, the main result of talk 6 only shows up in talk 11, but you only need to know that it suffices to prove (4). The seminar is going to end with another research talk by Henri Johnston.

Talks

1. **Talk by Samit Dasgupta** **November 5**
   Samit Dasgupta will give an online talk in the 'ESAGA Oberseminar' of our research groups in Essen:
   https://www.esaga.uni-due.de/ws2021/oberseminar/

   I'll send you the required login details in due course.

   I have asked him to give a talk on his work with Mahesh Kakde on the Brumer–Stark conjecture so that this talk will serve as an introduction to the topic of the seminar.

2. **Cusps** **November 10**
   The main reference for this talk is [9]. Define the action of $\mathrm{GL}_2^+(\mathbb{R})^n$ on the product of n upper half-planes and explain the notion of cusps for discrete

subgroups Γ of $\mathrm{SL}_2(\mathbb{R})^n$. You should do this first in the case $n = 1$ [9, bottom of page 11–13] and then for general n [9, §2, bottom of page 21 to Definition 2.7].

As the definition of a congruence subgroup we take that of [10, §1.1, p. 3]. State also the subsequent obvious proposition. In particular, if $\mathcal{O}$ is the ring of integers in a totally real number field K, then the 'Hilbert modular group' $\mathrm{SL}_2(\mathcal{O})$ is a congruence subgroup. Determine the cusps of $\mathrm{SL}_2(\mathcal{O})$ [9, Proposition 3.4] and show that the cusp classes under the action of this group are in one-to-one correspondence with the class group of K ([9, Lemma 3.5] or [10, Theorem on p. 7]). Deduce that there are only finitely many cusp classes for any congruence subgroup [10, first Corollary on p. 8] (the claim about the disjoint union is also used in the next talk!). If time permits, explain the statement of [9, Theorem 3.6].

3. **Hilbert modular forms** **November 17**
   Introduce the notion and first properties of (weak) Hilbert modular forms and cuspforms. For this, present the material of [10, §1.2]. State also the second Corollary on page 8. Note that by 'Hilbert modular form' we shall always mean 'holomorphic Hilbert modular form. State the Theorem on p. 23 (the space of cusp forms is finite dimensional) without proof.

   Give a proof of the 'Koecher principle' [10, §1.4] and deduce that we won't need to distinguish between weak and non-weak Hilbert modular forms in the seminar as we are not really interested in the case $K = \mathbb{Q}$.

   Let $\mathbb{A}$ be the adeles of K. Actually, it would be more natural to work with automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ rather than with 'classical' Hilbert modular forms. Since Dasgupta and Kakde stick to the classical notion, we will only sketch the relation in order to understand, why in the paper a Hilbert modular form is actually a tuple of forms in the above sense, and why the choices of congruence subgroups make sense. First assume that the narrow class group of K is trivial. Then it is explained in [10, §3.1, pp. 86–91], how a Hilbert modular form gives rise to a function on $\mathrm{GL}_2(\mathbb{A})$ that satisfies certain properties (the converse is also true). The general case is then outlined on pp. 92–93. These results often require tedious computations that you may wish to skip. Unfortunately, the congruence subgroups used in the book are not quite the same as in [6], but the idea should become clear.

4. **Preliminaries** **November 24**
   In this talk you should cover the contents of §2.1 to §3.3.

   More precisely, recall the definition of Artin L-functions (for linear characters) and the formula for its order of vanishing at $s = 0$ (see [18, Chapitre I, Proposition 3.4], for instance). Define Stickelberger elements and deduce that these vanish unless K is totally real and L is totally complex. Quote (without proof) that we can assume that L is CM (i.e. for the verification of the Brumer–Stark conjecture away from 2). Prove Lemma 2.1. Introduce character group rings.

Recall the definition of (initial) Fitting ideals and its basic properties (and those discussed in §2.3). Introduce the Selmer module of Burns, Kurihara and Sano (see also [3]). State the 'keystone result' (Theorem 3.3) and show, how Theorem 1 follows from this. If you restrict yourself to this implication (rather than presenting the stronger statement that Theorem 3.3 implies Kurihara's conjecture), the proof should simplify considerably.

5. **Some reduction steps and the Ritter–Weiss module December 1**
You should cover §4, §7.1 and §6 in this talk. First present the two reduction steps in §4 and the reduction step in §7.1.

   Then introduce the notion of the transpose and state Lemma 6.1. Construct the Ritter–Weiss module ∇. This is done in the appendix A.1 – A.5. You may not give all, but as many details as possible. You may also consult the original article by Ritter and Weiss [13]. Moreover, I have also some handwritten notes on constructing Tate sequences with the Ritter–Weiss method. I am happy to share them with you.

6. **Divisibility implies equality December 8**
Prove Theorem 5.1, i.e. present the contents of §5.

7. **Group ring valued Hilbert modular forms and Eisenstein series**

   December 15

   This talk covers [6, §7.2 and §7.3]. Explain how Dasgupta and Kakde define Hilbert modular forms. It should be clear from talk 3 that this definition is reasonable. Talk about q-expansions and forms with nebentypus before you introduce the Hecke operators (most of this is known from the previous talks, but we also need to setup notation). Since a Hilbert modular form is determined by its (normalized) Fourier coefficients, we can define the Hecke operators by their impact on these coefficients (there is, of course, a more conceptual way to define Hecke operators; if you are curious you can consult [14, p. 648], for instance). You may find the definition of the 'diamond operators' in [7, §2.5]. Introduce group ring valued Hilbert modular forms and explain the action of the Hecke operators $T_{\mathfrak{q}}$ on them [6, (97)]. From this description it should be also clear that the Eisenstein series (which you should define as well) are eigenvectors for these Hecke operators. It actually might be worth introducing the slightly more general Eisenstein series $E_k(\chi_1, \chi_2)$ (see [5, Proposition 2.1], for instance). When you introduce Hida's ordinary operator, you may give a proof of [11, §7.2, Lemma 1]. If you have time, it might be also worth to motivate the definition of Eisenstein series and Hecke operators by first considering the classical situation ($K = \mathbb{Q}$ and ψ a Dirichlet character). A good exposition is the introduction of Don Zagier in [2] (see §2.1 and §2.2).

8. **Construction of cusp forms I January 12**
This covers §8.1 and §8.2. Introduce the modified Eisenstein series W_k (Definition 8.2). We have to compute their constant terms at all cusps. State

Proposition 8.4 which is taken from [7]. It is then clear that one can compute the cusps of W_k for each $k \geq 1$, but it is not worth to state the precise result (at least not in the case $k = 1$) as nobody is able to remember these formulas. Present the material of §8.2 which includes two results of Silliman [16] that we take for granted. However, if time permits, you may give a hint, how one of these results is established or how one computes the constant terms of Eisenstein series. For instance, you may prove (or sketch a proof of) a particular case (e.g. [7, Theorem 4.5 (1)]).

9. **Construction of cusp forms II** **January 19**
There is not much to say about the following talks. Here you should cover §8.3 to §8.5. Theorem 8.23 is essential.

10. **Construction of a cohomology class** **January 26**
Theorem 8.23 and the theory of Wiles and Hida give rise to the cohomology class κ mentioned above. Cover §9.1 to §9.3 and Proposition 9.5.

11. **Talk by Henri Johnston** **February 2**
Henri Johnston will give a talk on a joint project with myself. As an application of the work of Dasgupta and Kakde we prove the abelian Iwasawa main conjecture for totally real fields and a strong form of the Coates–Sinnott conjecture on the annihilation of higher K-groups of rings of integers (away from 2).

12. **Finale: Proof of Brumer–Stark** **February 9**
Finish the proof of the Brumer–Stark conjecture by presenting the rest of §9.4 and calculating the required Fitting ideal (§9.5).

References

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