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10. CONSTRUCTION OF A COHOMOLOGY CLASS

§ 9.0 OVERVIEW

GOAL: $\mathrm{Fitt}_p(\mathrm{Sel}_{\Sigma}^{\Sigma'}(H)_p) \subseteq (\Theta_{\Sigma,\Sigma'}^{\#})$; maps

$\mathbb{T}$ Hecke algebra, B $\mathbb{T}$-module, $I \subseteq \mathbb{T}$

GOAL: build $\kappa \in H^1(G_F, "B/IB")$ with certain properties

$\mathrm{CFT} \rightsquigarrow A_L^{\mathbb{T}} \twoheadrightarrow B/IB$
$(\mathrm{Sel}\ldots)$

$$\mathrm{Fitt}_{\mathbb{Z}_p[G]^-}((A_L^{\mathbb{T}})) \subseteq \mathrm{Fitt}_{\mathbb{Z}_p[G]^-}(B/IB) \subseteq (\Theta^{\#})$$

Sel

$\sigma \longmapsto b(\sigma)$ (correction...)

$$\rho: G_F \longrightarrow GL_2\Big(\prod_{\overline{M}} E\Big)$$

$$\sigma \longmapsto \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$$

$\| \prod_{f \in \overline{M}} \rho_f$

§9.$\frac{1}{2}$ NOTATION AND MAPS

H/F abelian extension of # fields

$G = Gal(H/F)$ $\qquad S \supseteq S_\infty \uplus S_{ram},\ T \quad T \cap S = \emptyset$

$\cong G_p \times G'$ $\qquad \{\zeta \in \mu(H) : \zeta \equiv 1 (\mathfrak{P})\ \forall \mathfrak{P} \in T_H\} = \{1\}$

T has no prime $|p$

$\Sigma = \{v \in S_{ram} : v | p\} \cup S_\infty$

$\Sigma' = \{v \in S_{ram} : v \nmid p\} \cup T$

χ totally odd faithful character of G'

R character group ring associated to $\Psi = \{\psi \in \hat{G} : \psi|_{G'} = \chi,\ \psi(v) \neq 1\ \forall v \in \Sigma\}$

$\boxed{\mathfrak{m}} = \mathrm{cond}(H/F) \prod_{\mathfrak{q} \in T} \mathfrak{q}$

$\boxed{\mathfrak{O}} = \gcd(p^\infty, \mathfrak{m})$

$\boxed{\mathfrak{O}'} = \prod_{\mathfrak{P}|p,\ \mathfrak{P} \nmid \mathfrak{O}} \mathfrak{P}$

$\boxed{\mathfrak{O}''} = \prod_{\substack{\mathfrak{P} | \mathfrak{O}' \\ \chi(\mathfrak{P}) \neq 1}} \mathfrak{P}$

G quotient of $G_\mathfrak{m}^+ \rightsquigarrow \boxed{\underline{\Psi}} : G_\mathfrak{m}^+ \twoheadrightarrow G \rightarrow R^\times$

$\boxed{k} \equiv 1 \ (\mathrm{mod}\ (p-1)p^N)$ for $N >> 0$

$\boxed{\tilde{\mathbb{T}}} \subseteq \mathrm{End}_R(S_{k}(\mathfrak{m}\mathfrak{O}', R, \underline{\Psi})^{p\text{-ord}})$ Hecke alg.

gen. by:
- $T_l \quad l \nmid \mathfrak{m}\mathfrak{O}'$
- $U_\mathfrak{P} \quad \mathfrak{P} | p$
- $S(m),\ m \in G_\mathfrak{m}^+ \qquad (S(m) \leftrightarrow \underline{\Psi}(m))$

$\boxed{M} := \{p$-ordinary cuspidal newforms of weight k, level $|\mathfrak{m}\mathfrak{O}'$, nebentypus $\psi\ \forall \psi \in \Psi\}$

$f \to f_p$ p-ordinary stabilisation $((1-p^{k-1}V_p)f$?) 3

$\mathcal{O} = \mathcal{O}_E$ $\boxed{E}/\mathbb{Q}_p$ big so that $c(\square, f_p) \in \mathcal{O}$ $\forall \square, f$

$\boxed{T} \subseteq \tilde{T}$ sub R-algebra gen. over R by:

- T_ℓ $\ell \nmid \mathfrak{n}\mathfrak{P}'$ (exclude $\mathfrak{p} \mid \mathfrak{P}'$)
- $U_\mathfrak{p}$ $\mathfrak{p} \mid \mathfrak{P}$
- $S(\mathfrak{m})$, $\mathfrak{m} \in G_\mathfrak{n}^+$

$$T \hookrightarrow \tilde{T} \to \prod_M \mathcal{O} \quad \text{with finite cokernel}$$
$$t \longmapsto (c(1, f_p)|_t)_{f \in M}$$

<u>LEMMA 8.22</u>: $\mathcal{O}$-algebra hom $\boxed{\bar{\varphi}} : T \to k_E$

$T_\ell \mapsto 1 + \chi(\ell)$ $\ell \nmid \mathfrak{n}p$

$U_\mathfrak{p} \mapsto 1$ $\mathfrak{p} \mid \mathfrak{P}$

$S(\mathfrak{m}) \mapsto \chi(\mathfrak{m})$ $\mathfrak{m} \in G_\mathfrak{n}^+$

$\boxed{\mathfrak{m}} \subseteq T$ is $\ker \bar{\varphi}$

$\boxed{T_\mathfrak{m}}$ $\mathfrak{m}$-adic compl. of T, $\tilde{T}_\mathfrak{m} = \tilde{T} \otimes_T T_\mathfrak{m}$

$\boxed{\bar{M}} := \{ f \in M : c(1, f_p)|_t \equiv \bar{\varphi}(t) \pmod{\pi_E}\ \forall t \in T \}$

$\subseteq M$

Then $\left(\prod_{f \in M} \mathcal{O}\right)_\mathfrak{m} = \prod_{f \in \bar{M}} \mathcal{O}$

[illegible]:
$$\begin{tikzcd} T_\mathfrak{m} \arrow[r] \arrow[rr, bend right] & \tilde{T}_\mathfrak{m} \arrow[r] & \prod_{f \in \bar{M}} \mathcal{O} \end{tikzcd} \quad \text{fin. cokernel}$$

$\boxed{\epsilon_{cyc}}(\ell) = \langle N\ell \rangle \in \mathbb{Z}_p^\times$

<u>TH 8.23</u>: $\exists\, x \in R$ non-zero divisor, $R/x\Theta^\#$-algebra $\boxed{W}$, $\varphi : \tilde{T}_\mathfrak{m} \to W$ s.t.

- $R/x\Theta^\# \hookrightarrow W$
- $\varphi(T_\mathfrak{m}) \subseteq R/x\Theta^\#$
- $S(\mathfrak{m}) \mapsto \underline{\varphi}(\mathfrak{m})$, $\mathfrak{m} \in G_\mathfrak{n}^+$; $U_\mathfrak{p} \mapsto 1$ $\mathfrak{p} \mid \mathfrak{P}$; $T_\ell \mapsto \epsilon_{cyc}^{-1}(\ell) + \underline{\varphi}(\ell)$ $\ell \nmid \mathfrak{n}p$
- $U := \varphi\left(\prod_{\mathfrak{p} \mid \mathfrak{P}'} (U_\mathfrak{p} - \underline{\psi}(\mathfrak{p}))\right)$, $y \in R$. Then $Uy = 0$ in $W \Rightarrow y \in (\Theta^\#)$

§9.1 GALOIS REPRESENTATION ASSOCIATED TO EACH EIGENFORM

$f \in \overline{M}$, $\psi^{6\Phi}$ nebentypus of f. Hida–Wiles:

cont. $\rightsquigarrow \rho_f : G_F \longrightarrow GL_2(E)$ s.t.

(i) ρ_f unram. outside $\mathfrak{n}p$

(ii) $l \nmid \mathfrak{n}p$ then $\mathrm{char}(\rho_f(\mathrm{Frob}_l))(x)$

$$= x^2 - c(l,f)x + \psi(l)\epsilon_{cyc}^{k-1}(l)$$

(iii) $\mathfrak{P} | p$

$$\rho_f|_{G_\mathfrak{P}} \sim \begin{pmatrix} \psi \eta_\mathfrak{P}^{-1} \epsilon_{cyc}^{k-1} & * \\ 0 & \eta_\mathfrak{P} \end{pmatrix}$$

$\eta_\mathfrak{P}$ unram. char. $rec(\varpi) \mapsto c(P, f_p)$

$V_{P,f} \hookleftarrow \mathrm{Span}\begin{pmatrix}1\\0\end{pmatrix}$

PROPERTIES: $\forall \sigma \in G_F$.

- (ii) + Chebotarev $\Longrightarrow \mathrm{char}(\rho_f(\sigma)) \in \mathcal{O}[x]$
- $\psi \equiv \chi \pmod{\pi_E}$, k big enough $\Longrightarrow$

$$\psi(l)\epsilon_{cyc}^{k-1}(l) \equiv \chi(l) \pmod{\pi_E}$$

$$c(l,f) \equiv c(l,f_p) \overset{f\in\overline{M}}{\equiv} \overline{\Phi}(T_l) \equiv 1 + \chi(l) \pmod{\pi_E}$$

$$\Longrightarrow \mathrm{char}(\rho_f(\sigma)) \equiv (x-1)(x-\chi(\sigma)) \pmod{\pi_E}$$

τ s.t. $\chi(\tau) \neq 1 \Leftrightarrow \chi(\bar{\tau}) \not\equiv 1 \pmod{\pi_\epsilon}$ ⑤

Hecke $\Rightarrow$ $\rho(\tau)$ has two eigenvalues

$$\lambda_{1,f} \equiv 1 (\pi_\epsilon)$$
$$\lambda_{2,f} \equiv \chi(\tau) \pmod{\pi_\epsilon}$$

DEF: f CM form if $\rho_f = \mathrm{Ind}_{G_L}^{G_F} \alpha$ where

L/F quadratic CM extension

α p-adic Hecke character of L

LEMMA 9.1: f cuspidal eigenform of weight $k > 1$, f is not CM. Then the restriction of ρ_f to any finite index subgroup of G_F is irreducible.

LEMMA 9.2: $f \in \bar{M}$ CM form ass. to L/F. P/p. Then for every $\tilde{c}/_L$ = complex conj. $V_{P,f}$ is not stable under $\rho_f(\tau)$

PROOF: f ordinary at $P \Rightarrow P$ splits in L/F ??

$G_P \subseteq G_L$. ρ_f has two ~~subgroups~~ subspaces stable under G_L. $\rho|_{G_P}$ has non. and unram. char. as eigenvalues $\Rightarrow V_{P,f}$ one of those. $V_{P,f}$ invariant under $\tau \Rightarrow$ invariant under $G_F \Rightarrow \rho_f$ reducible. ⚡

PROP 9.3 : $\exists \tau \in G_F$ s.t. $\tau|_L$ complex conj. and $\forall f \in \bar{M}$ and P/p $\quad$ (6)

$V_{P,f}$ not stable under $\rho_f(\tau)$.

PROOF : $H_0 = H \cdot \{$CM fields L as. $f \in \bar{M}$ (CM)$\}$

on H_0/F OK $\forall$ CM $f \in \bar{M}$ and P/p

$V_1, \dots, V_n$ be all the remaining $V_{P,f}$

G_i = stab. of V_i under ρ_f. $[G_F : G_i] = +\infty$

$\Rightarrow \exists\, a_i \in \bar{F}$ fixed by G_i and not H_{i-1}.

$\rightsquigarrow H_i$ = Gal. closure of $H_{i-1}(a_i)$.

$\tau_i \in \mathrm{Gal}(H_i/F)$ restricts to τ_{i-1} on H_{i-1} s.t. $\tau_i(a_i) \neq a_i$.

take τ restricting to τ_i $\forall i$ on H_i.

moves $a_i \Rightarrow \tau \notin G_i \Rightarrow \tau$ doesn't stabilize V_i under $\rho_f \Rightarrow \tau$ restr. τ_n does the job. □

$\forall \ell$ choose basis s.t.

$$\rho_\ell(\tau) = \begin{pmatrix} \lambda_{1,\ell} & 0 \\ 0 & \lambda_{2,\ell} \end{pmatrix} \qquad \begin{array}{l} \lambda_{1,\ell} \equiv 1 \,(\pi_E) \\ \lambda_{2,\ell} \equiv \chi(\tau) \equiv -1 \,(\pi_E) \end{array}$$

$$\rho_\ell(\sigma) = \begin{pmatrix} a_\ell(\sigma) & b_\ell(\sigma) \\ c_\ell(\sigma) & d_\ell(\sigma) \end{pmatrix}$$

$$M_{\ell,P} = \begin{pmatrix} A_{\ell,P} & B_{\ell,P} \\ C_{\ell,P} & D_{\ell,P} \end{pmatrix} \in GL_2(E) \quad \text{s.t.}$$

$$\begin{pmatrix} a_\ell(\sigma) & b_\ell(\sigma) \\ c_\ell(\sigma) & d_\ell(\sigma) \end{pmatrix} M_{\ell,P} = M_{\ell,P} \begin{pmatrix} \psi \eta_P^{-1} \varepsilon_{cyc}^{k-1}(\sigma) & * \\ 0 & \eta_P(\sigma) \end{pmatrix}$$

$$\rightsquigarrow \forall \sigma \in G_P : \forall \ell \in \bar{m}, P \mid \ell \quad \begin{array}{l} A_{\ell,P} \neq 0 \\ C_{\ell,P} \neq 0 \end{array}$$

§9.2 GALOIS REPRESENTATIONS ASSOCIATED TO T_m

$$K := \mathrm{Frac}(T_m) = \mathrm{Frac}\Big(\prod_{f \in \tilde{m}} \mathcal{O}\Big) = \prod_{f \in \tilde{m}} E$$

Gal rep.: $\rho = \prod_{f \in \tilde{m}} \rho_f : G_F \longrightarrow GL_2(K)$

$\wr$ unit. rep. p-adic $\Rightarrow$ rep. m-adic

(i) unram. outside mp

(ii) $l \nmid mp$ $\quad \mathrm{char}(\rho(\mathrm{Frob}_l))(x) = x^2 - T_l x + \Psi(l)\, \epsilon_{cyc}^{k_2-1}$

(iii) $\mathfrak{P} | p$ $\quad \rho|_{G_{\mathfrak{P}}} \sim \begin{pmatrix} \Psi\, \eta_{\mathfrak{P}}^{-1} \epsilon_{cyc}^{k_2-1} & * \\ 0 & \eta_{\mathfrak{P}} \end{pmatrix}$

$\eta_{\mathfrak{P}} : G_{\mathfrak{P}} \longrightarrow \tilde{T}^\times$ unram. given by

$\mathrm{rec}(\varpi) \longmapsto U_{\mathfrak{P}}$

- Chebotarev $\Rightarrow$ $\mathrm{char}(\rho(\sigma))(x) \in T_m[x] \quad \forall \sigma \in G_F$
- $\varphi(T_l) = \epsilon_{cyc}^{k-1}(l) + \Psi(l) \Rightarrow \mathrm{char}\, \rho(\sigma)(x) \equiv (x-1)(x - \chi(\sigma)) \pmod{m}$

τ as before $\Rightarrow$ $\mathrm{char}(\rho(\tau))$ two distinct roots mod m

$\xRightarrow{\text{Hensel}}$ two distinct roots $\lambda_1, \lambda_2 \in T_m^\times$: $\lambda_1 \equiv 1 \,(m)$
$\lambda_2 \equiv -1 \,(m)$

choose basis s.t. $\rho(\tau) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\rho(\sigma) =: \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$$

$A_{\mathfrak{P}} = (A_{f,\mathfrak{P}})_{f \in \tilde{m}}$ etc.

$\forall \mathfrak{P} | p$ $\quad M_{\mathfrak{P}} = \begin{pmatrix} A_{\mathfrak{P}} & B_{\mathfrak{P}} \\ C_{\mathfrak{P}} & D_{\mathfrak{P}} \end{pmatrix} \in GL_2(K)$ s.t. $\quad A_{\mathfrak{P}}, C_{\mathfrak{P}} \in K^\times$

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} M_{\mathfrak{P}} = M_{\mathfrak{P}} \begin{pmatrix} \Psi\, \eta_{\mathfrak{P}}^{-1} \epsilon_{cyc}^{k-1}(\sigma) & * \\ 0 & \eta_{\mathfrak{P}}(\sigma) \end{pmatrix} \quad \forall \sigma \in G_{\mathfrak{P}}$$

we get $\boxed{b(\sigma) = \frac{A_p}{C_p}\left(\Psi\, u_p^{-1}\, \epsilon_{cyc}^{k-1}(\sigma) - a(\sigma)\right)}$

§§ 3 COHOMOLOGY CLASS AND RAMIFICATION OUTSIDE p

$$\varphi : \tilde{T}_m \to W \qquad \tilde{I} := \ker \varphi$$
$$\tilde{T}_m \supseteq T_m \qquad I := \hat{I} \cap T_m$$

GOAL: understand $\lambda_1, \lambda_2 \bmod I$

$$\left.\begin{array}{l}\varphi(T_\ell) = \epsilon_{cyc}^{k-1}(\ell) + \Psi(\ell) \\ \quad \| \\ \varphi(\operatorname{tr}\rho(\mathrm{Frob}_\ell))\end{array}\right\} \Rightarrow \begin{array}{l} a(\sigma) + d(\sigma) \equiv \epsilon_{cyc}^{k-1}(\sigma) + \Psi(\sigma)\ (I) \\ \lambda_1 + \lambda_2 \equiv \epsilon_{cyc}^{k-1}(\tau) + \Psi(\tau) \pmod I\end{array}$$

$\Rightarrow \lambda_1, \lambda_2$ roots of $(x - \epsilon_{cyc}^{k-1}(\tau))(x - \Psi(\tau)) \pmod I$

$$?\ \left|\ \begin{array}{l}\lambda_1 \equiv 1 \equiv \epsilon_{cyc}^{k-1}(\tau) \pmod m \\ \not\equiv \\ \lambda_2 \equiv \chi(\tau) \equiv \Psi(\tau) \pmod m\end{array}\right\} \Rightarrow \begin{array}{l}\lambda_1 \equiv \epsilon_{cyc}^{k-1}(\tau)\ (I) \\ \lambda_2 \equiv \Psi(\tau)\ (I)\end{array}$$

$$a(\sigma)\lambda_1 + d(\sigma)\lambda_2 \equiv \epsilon_{cyc}^{k-1}(\sigma\tau) + \Psi(\sigma\tau) \equiv$$
$$\equiv \epsilon_{cyc}^{k-1}(\sigma)\lambda_1 + \Psi(\sigma)\lambda_2$$

$$\lambda_1 \not\equiv \lambda_2 (m) \Rightarrow \begin{array}{l} a(\sigma) \equiv \epsilon_{cyc}^{k-1}(\sigma)\ (I) \\ d(\sigma) \equiv \Psi(\sigma)\ (I)\end{array} \quad \forall \sigma \in G_F$$

(9)

B_0 T_m- submodule of K gen. by $\{ b(\sigma), \sigma \in G_F \}$

$B \supseteq B' \supseteq I B_0$ $\quad \bar{B} := B/_{B'}$

$\rightsquigarrow b(\sigma\sigma') = a(\sigma) b(\sigma') + b(\sigma) d(\sigma')$

$\Rightarrow \bar{b}(\sigma\sigma') = \epsilon_{cy}^{k-1}(\sigma) \bar{b}(\sigma') + \underline{\Psi}(\sigma') \bar{b}(\sigma)$ in $\bar{B}$

So $\sigma \longmapsto \bar{b}(\sigma) \underline{\Psi}(\sigma)^{-1}$ is a 1-cocycle

$\left(\kappa \in H^1(G_F, \bar{B}(\underline{\Psi}^{-1} \epsilon_{cy}^{k-1})) \right)$

$p^m B \subseteq B'$ and $k \equiv 1 (\mathrm{mod}(p-1) p^N))$ $N \geq m$

$\rightsquigarrow \kappa \in H^1(G_F, \bar{B}(\underline{\Psi}^{-1}))$.

PROP 9.1: κ is unramified away from $\mathfrak{n}p$, i.e. image in $H^1(I_v, \bar{B}(\underline{\Psi}^{-1}))$ is 0. $\forall v \nmid \mathfrak{n}p$.

For $v | \mathfrak{n}$, $v \nmid p \rightsquigarrow \kappa$ is at most tamely ramified: restriction to $I_{v,1} \subset I_v$ vanishes.

PROOF: ρ unram. outside $\mathfrak{n}p \Rightarrow b(\sigma) = 0$ $\forall \sigma \in I_v$ $v \nmid \mathfrak{n}p$.

$I_{v,1}$ pro-v group, $\bar{B}$ pro p-group $\Rightarrow$

$\Rightarrow H^1(I_{v,1}, \bar{B}(\underline{\Psi}^{-1})) = 0$ $\forall v \nmid p$. $\square$

§ 9. A SURJECTION FROM ∇

B_0 : $\mathbb{T}_{\mathfrak{m}}$ - submodule of K gen. by $b(\sigma), \sigma \in G_F$

B gen. by B_0 and $\left\{ A_\mathfrak{P} / C_\mathfrak{P} \right\}_{\mathfrak{P} \in \Sigma \setminus S_\infty}$

$\cup\!\!|$

B' gen. by $b(\sigma)$,

$\sigma \in I_\mathfrak{P}$ $\mathfrak{P} \mid \mathcal{O}'$

$\Sigma = \{ v \in S_{ram} \ v \mid p \} \cup S_\infty$

$\Sigma' = \{ v \in S_{ram} : v \nmid p \} \cup T$

$$\overline{B}_p = B / (IB, B', p^m B)$$

$\cup\!\!|$

$\overline{B}_0$ image of B_0

PROP 9.5: $K \in H^1(G_F, \overline{B}_0(\Psi^{-1}))$

unramified away from Σ', tamely ram. at Σ', locally trivial at Σ.

PROOF: unram. away from $\mathfrak{N}p \Rightarrow$ unram. away from $\Sigma \cup \Sigma' \cup \{ \mathfrak{P} \mid p \}$

tamely ram. at $v \mid \mathfrak{N}$, $v \nmid p \Rightarrow$ tamely ram. at Σ'

$\mathfrak{P} \mid p$ not in Σ, $I_\mathfrak{P}$ goes to $0 \in \overline{B}_p$ by def.

p odd $\Rightarrow$ locally trivial at S_∞.

locally trivial at $\mathfrak{P} \in \Sigma$ finite:

$$\left(\mathcal{A}_\mathfrak{p} / C_\mathfrak{p}\right)_{B}^{\wedge} I = 0 \quad \text{in } \bar{B}.$$

$$\varphi(v_\mathfrak{p}) = 1 \Rightarrow \eta_\mathfrak{p}(\mathrm{rec}(\bar{\omega})) \equiv 1 \pmod{I}$$
$$\Rightarrow \eta_\mathfrak{p} \equiv 1 \ (I) \quad \forall \mathfrak{p} \in S.$$

Here: $\Sigma \neq S_\infty$ (case 2) $\xrightarrow[\text{8.23}]{\text{proof of}}$ $\epsilon_{cyc}^{\kappa-1} \equiv 1 \pmod{I} \Rightarrow$

$$\Rightarrow a(\sigma) \equiv 1 \ (I).$$

$$b(\sigma) = \frac{\mathcal{A}_\mathfrak{p}}{C_\mathfrak{p}}\left(\Psi \eta_\mathfrak{p}^{-1} \epsilon_{cyc}^{\kappa-1}(\sigma) - a(\sigma)\right)$$

$$\Rightarrow b(\sigma)\Psi^{-1}(\sigma) \equiv \left(1 - \Psi^{-1}(\sigma)\right)\frac{\mathcal{A}_\mathfrak{p}}{C_\mathfrak{p}} \quad \text{in } \bar{B}$$

for $\sigma \in G_\mathfrak{p}$.

$\rightsquigarrow K|_{G_\mathfrak{p}}$ is coboundary. $\square$