An unconditional proof of the abelian equivariant Iwasawa main conjecture

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https://arxiv.org/abs/2010.03186

Tuesday 2nd February 2021
Essener Seminar für Algebraische Geometrie und Arithmetik

The idea of the EIMC and why we care

Let p be an odd prime and K be a totally real number field.
Let $\mathcal{L}/K$ be an 'admissible' an extension.

Roughly speaking, the EIMC relates an Iwasawa module attached to $\mathcal{L}/K$ to Artin L-functions via p-adic L-functions.

The EIMC can be phrased as asserting the *existence* of an element with certain properties in an algebraic K-group; it is also conjectured that this element is *unique*.

Why do we care?

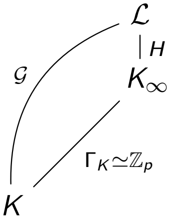
- The EIMC can be used to prove explicit annihilation results involving special values of Artin L-functions (e.g. Brumer-Stark and Coates-Sinnott conjectures).
- The EIMC is an important ingredient in the proof of many cases of the ETNC for Galois extensions of number fields.
- The EIMC is the 'easiest' case of a much more general conjecture that can be formulated for other motives.

Admissible extensions

Let p be an odd prime and let K be a totally real number field.

Let $\mathcal{L}$ be a Galois extension of K such that
(i) $\mathcal{L}$ is totally real,
(ii) $\mathcal{L}$ contains the cyclotomic $\mathbb{Z}_p$-extension K_∞ of K, and
(iii) $[\mathcal{L} : K_\infty]$ is finite.

$$\begin{aligned} H &= \mathrm{Gal}(\mathcal{L}/K_\infty) \\ \Gamma_K &= \mathrm{Gal}(K_\infty/K) \simeq \mathbb{Z}_p \\ \mathcal{G} &= \mathrm{Gal}(\mathcal{L}/K) = H \rtimes \Gamma \\ &\text{where } \Gamma \simeq \Gamma_K \simeq \mathbb{Z}_p \end{aligned}$$



We say $\mathcal{L}/K$ is a *one-dimensional admissible extension*.

Note if L/K is a finite Galois extension and is L totally real, then L_∞/K is such an extension.

Can also define arbitrary admissible extensions.

The Iwasawa algebra $\Lambda(\mathcal{G})$

Define the *Iwasawa algebra* of $\mathcal{G}$ to be

$$\Lambda(\mathcal{G}) := \mathbb{Z}_p[\![\mathcal{G}]\!] = \varprojlim \mathbb{Z}_p[\mathcal{G}/\mathcal{N}],$$

where $\mathcal{N}$ ranges over all open normal subgroups of $\mathcal{G}$.

Note that as H is finite, $\mathrm{Aut}(H)$ is also finite, and so there exists an open subgroup $\Gamma_0 \leq \Gamma$ that is central in $\mathcal{G} = H \rtimes \Gamma$.

Let $R = \mathbb{Z}_p[\![\Gamma_0]\!]$. Then $\Lambda(\mathcal{G})$ is an R-algebra and is free of finite rank as an R-module.

Note that $R \simeq \mathbb{Z}_p[\![T]\!]$ is a regular local ring and thus is of finite global dimension.

However, $\Lambda(\mathcal{G})$ is of finite global dimension if and only if $p \nmid |H|$.

The Iwasawa module X_S

Let S be a finite set of primes of K s.t. $S_\infty \cup S_{\mathrm{ram}}(\mathcal{L}/K) \subseteq S$. Note that $S_p \subseteq S_{\mathrm{ram}}(\mathcal{L}/K)$.

Let $M_S^{\mathrm{ab}}(p)$ be the maximal abelian pro-p-extension of $\mathcal{L}$ unramified outside S and let $X_S = \mathrm{Gal}(M_S^{\mathrm{ab}}(p)/\mathcal{L})$.

Since $K \subseteq \mathcal{L} \subseteq M_S^{\mathrm{ab}}(p)$ there is a canonical short exact sequence

$$1 \longrightarrow X_S \longrightarrow \mathrm{Gal}(M_S^{\mathrm{ab}}(p)/K) \longrightarrow \mathcal{G} \longrightarrow 1,$$

which defines an action of $\mathcal{G}$ on X_S. So X_S is a module over $\Lambda(\mathcal{G})$.

This is the Iwasawa module used in the formulation of the EIMC.

Note that the truth of the EIMC is independent of the choice of S.

The projective dimension of X_S and the complex $C_S^\bullet(\mathcal{L}/K)$

As an R-module, X_S is finitely generated, torsion and of finite projective dimension.

If $p \nmid |H|$ then X_S is also of finite projective dimension over $\Lambda(\mathcal{G})$, and so the EIMC can be stated directly in terms of X_S.

In general, X_S is *not* of finite projective dimension over $\Lambda(\mathcal{G})$, so one has to replace X_S by a certain complex $C_S^\bullet(\mathcal{L}/K)$ of $\Lambda(\mathcal{G})$-modules which is perfect and whose only non-vanishing cohomology modules are isomorphic to X_S and $\mathbb{Z}_p$, respectively.

The $\mu = 0$ hypothesis

We say that an admissible extension $\mathcal{L}/K$ satisfies the $\mu = 0$ hypothesis if X_S is finitely generated as a $\mathbb{Z}_p$-module.

This is independent of the choice of S.

From work of Ferrero and Washington, one can deduce that the $\mu = 0$ hypothesis holds whenever $\mathcal{L}/K$ is an admissible extension such that $\mathcal{L}$ is a pro-p extension of a finite abelian extension of $\mathbb{Q}$.

Unfortunately, little is known beyond this case.

Different versions of the EIMC

- Ritter and Weiss (JAMS, 2011)
  Formulated version for one-dimensional admissible extensions and showed it holds when $\mu = 0$.
- Kakde (Inventiones, 2013)
  Proved a version for arbitrary admissible extensions and showed it holds when $\mu = 0$. Used strategy of Burns and Kato to reduce to the one-dimensional case.
- Greither and Popescu (J Alg Geom, 2015)
  Formulated a version for abelian one-dimensional extensions assuming $\mu = 0$, and also proved it in this situation.
- Nickel (Proc LMS, 2013)
  Generalised work of Greither and Popescu to non-abelian one-dimensional case.

Work of Nickel (and of Venjakob) showed that any two of these are equivalent when this makes sense. A key point is that complexes used are isomorphic in the derived category of $\Lambda(\mathcal{G})$-modules.

Known cases of the one-dimensional EIMC

Theorem (Ritter and Weiss, Kakde)

If $\mathcal{L}/K$ satisfies $\mu = 0$ then the EIMC holds for $\mathcal{L}/K$.

Theorem (Nickel, J - AJM 2018)

If $p \nmid |H|$ then the EIMC with uniqueness holds for $\mathcal{L}/K$.

Theorem (Nickel, J - AJM 2018)

If there exists a finite normal subgroup N of $\mathcal{G}$ such that $\Lambda(\mathcal{G}) \simeq \Lambda(\mathcal{G}/N) \times \mathcal{M}$ for some maximal R-order $\mathcal{M}$ then the EIMC for $\mathcal{L}/K$ holds if and only if it holds for $\mathcal{L}^N/K$.

From this last result one can deduce that the EIMC holds in certain special cases in which $\mu = 0$ is not known.

The main results (Nickel, J - preprint 2020)

Theorem

If $\mathcal{L}/K$ is an admissible abelian one-dimensional extension then the EIMC with uniqueness holds for $\mathcal{L}/K$.

Corollary

If $\mathcal{L}/K$ is an admissible one-dimensional extension such that $\mathcal{G}$ has an abelian Sylow p-subgroup then the EIMC with uniqueness holds for $\mathcal{L}/K$.

Note if we assume Leopoldt's conjecture then then 'one-dimensional' hypothesis in these results is superfluous.

Corollary

The Coates-Sinnott conjecture holds away from its 2-part.

Artin L-functions

Let L/K be a finite abelian extension of number fields with K totally real and L a CM field. Let $G = \mathrm{Gal}(L/K)$.

Let S, T be finite sets of places of K such that
(i) $S \supseteq S_\infty \cup S_{\mathrm{ram}}(L/K)$ and (ii) $S \cap T = \emptyset$.

For $\chi \in \hat{G}$ define the S-truncated Artin L-function attached to χ to be the meromorphic continuation to $\mathbb{C}$ of the holomorphic function

$$L_S(s,\chi) = \prod_{\mathfrak{p} \notin S} (1 - \chi(\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-s})^{-1} \quad \text{for} \quad \mathrm{Re}(s) > 1.$$

Define the (S, T)-modified Artin L-function to be

$$L_{S,T}(s,\chi) = \prod_{\mathfrak{q} \in T} (1 - \chi(\mathfrak{q})(\mathrm{N}\mathfrak{q})^{1-s}) L_S(s,\chi).$$

Stickelberger elements and integrality

Using the identification $\mathbb{C}[G] \cong \prod_{\chi \in \hat{G}} \mathbb{C}$, define

$$(\theta_S^T)^\# = (L_{S,T}(0,\chi))_{\chi \in \hat{G}} \in \mathbb{C}[G].$$

In fact, by a result of Siegel, Klingen and Shintani $(\theta_S^T)^\# \in \mathbb{Q}[G]$.

Now suppose (iii) the group of roots of unity $\zeta \in \mu(L)$ such that $\zeta \equiv 1 \bmod \mathfrak{P}$ for all $\mathfrak{P} \in T(L)$ is trivial.

By results of Pi. Cassou-Noguès and of Deligne and Ribet, we have

$$(\theta_S^T)^\# \in \mathbb{Z}[G].$$

The strong Brumer-Stark conjecture

Let cl_L^T denote the ray class group of L associated to $\prod_{\mathfrak{P} \in T(L)} \mathfrak{P}$. Let $(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{cl}_L^T)^{\vee}$ denote the Pontryagin dual of its p-part.

Theorem (Dasgupta and Kakde, preprint 2020)

Let p be an odd prime and suppose S, T satisfy (i), (ii) and (iii). Then

$$(\theta_S^T)^{\#} \in \mathrm{Fitt}_{\mathbb{Z}_p[G]}((\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{cl}_L^T)^{\vee}).$$

This is the key ingredient in our proof of the abelian EIMC.

In fact, we only need the weaker 'imprimitive' version of this result, in which S is assume to contain S_p, all the places of K above p.

Moreover, Dasgupta and Kakde actually prove something somewhat stronger than the above theorem.

The EIMC implies imprimitive strong Brumer-Stark

Nickel (AIF, 2011) formulated a non-abelian version of the strong Brumer-Stark conjecture. Independently, Burns (Inventiones, 2011) also formulated a non-abelian version in even greater generality.

(i) Greither and Popescu (J Alg Geom, 2015)
Showed that $\mu_p = 0$ implies the imprimitive strong Brumer-Stark conjecture.

(ii) Nickel (Proc LMS, 2013)
Showed that $\mu_p = 0$ implies the non-abelian imprimitive strong Brumer-Stark conjecture.

(iii) Nickel, J (Math Zeit, 2019)
Showed that the EIMC implies the non-abelian imprimitive strong Brumer-Stark conjecture. Does *not* assume $\mu_p = 0$.

Note (i) and (ii) require $\mu_p = 0$ not only so that the EIMC holds, but also for the descent from the EIMC to the finite level.

Outline of proof of the abelian EIMC

Idea is to reverse the proof of (iii) specialised to the abelian case.

Let $\mathcal{L}/K$ be a one-dimensional abelian admissible extension. Then there exists a finite abelian CM extension L/K such that (a) $\zeta_p \in L$, (b) $L \cap K_\infty = K$ and (c) $\mathcal{L} \subseteq L_\infty^+$.

Let S, T be finite sets of places of K satisfying the usual hypotheses and $S_p \subseteq S$. Let L_n denote the nth layer of L_∞/L.

Then $\mathrm{SBS}(L_n/K, S, T, p)$ for all n implies the EIMC for L_∞^+/K.

Note the EIMC for L_∞^+/K implies the EIMC for $\mathcal{L}/K$ by fuctoriality.

Statement of the EIMC

Let $\mathcal{Q}(\mathcal{G})$ be the total ring of fractions of $\Lambda(\mathcal{G})$.

There is an exact sequence of algebraic K-groups

$$K_1(\Lambda(\mathcal{G})) \longrightarrow K_1(\mathcal{Q}(\mathcal{G})) \xrightarrow{\partial} K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G})) \longrightarrow 0.$$

Since $C_S^\bullet(\mathcal{L}/K)$ is perfect and its non-vanishing cohomology modules X_S and $\mathbb{Z}_p$ are R-torsion, it defines an element

$$[C_S^\bullet(\mathcal{L}/K)] \in K_0(\Lambda(\mathcal{G}), \mathcal{Q}(\mathcal{G})).$$

There exists an element $\Phi_S(\mathcal{L}/K) \in Z(\mathcal{Q}(\mathcal{G}))^\times$ that encodes power series related to certain p-adic L-functions. Note that the reduced norm map induces a map $\mathrm{nr} : K_1(\mathcal{Q}(\mathcal{G})) \longrightarrow Z(\mathcal{Q}(\mathcal{G}))^\times$.

Conjecture (The EIMC with uniqueness)

There exists a unique $\zeta_S \in K_1(\mathcal{Q}(\mathcal{G}))$ such that $\mathrm{nr}(\zeta_S) = \Phi_S$. Moreover, $\partial(\zeta_S) = -[C_S^\bullet(\mathcal{L}/K)]$.

The EIMC in the abelian case

Now suppose $\mathcal{G}$ is abelian. Then $\mathrm{nr} : K_1(\mathcal{Q}(\mathcal{G})) \longrightarrow Z(\mathcal{Q}(\mathcal{G}))^{\times}$ is an isomorphism. Thus we automatically have uniqueness.

There exists a unique maximal R-order $\mathcal{M}(\mathcal{G})$ such that $\Lambda(\mathcal{G}) \subseteq \mathcal{M}(\mathcal{G}) \subseteq \mathcal{Q}(\mathcal{Q})$. Note that $\mathcal{M}(\mathcal{G})$ is a direct product of regular local rings and thus has finite global dimension.

The Iwasawa Main Conjecture (IMC) roughly speaking is equivalent to the abelian EIMC but with $\Lambda(\mathcal{G})$ replaced by $\mathcal{M}(\mathcal{G})$. The IMC was proven by Wiles (Annals 1990) without any $\mu = 0$ hypothesis.

Let $\mathfrak{q}$ be a height one prime ideal of $\Lambda(\mathcal{G})$.
If $\mathfrak{q}$ is *regular* (i.e. $p \notin \mathfrak{q}$) then $\Lambda(\mathcal{G})_{\mathfrak{q}} = \mathcal{M}(\mathcal{G})_{\mathfrak{q}}$. Thus we know the abelian EIMC after localisation at such primes $\mathfrak{q}$ thanks to the IMC. It remains to show the EIMC after localisation at height one *singular* primes (i.e. $p \in \mathfrak{q}$).

How to prove the abelian EIMC when $\mu = 0$

Note that since $\mathcal{M}(\mathcal{G})$ has finite global dimension we have that

$$[\mathcal{M}(\mathcal{G}) \otimes^{\mathbb{L}}_{\Lambda(\mathcal{G})} C^\bullet_S(\mathcal{L}/K)] \in K_0(\mathcal{M}(\mathcal{G}), \mathcal{Q}(\mathcal{G}))$$

is determined only by the cohomology of $C^\bullet_S(\mathcal{L}/K)$.

Since X_S and $\mathbb{Z}_p$ are finitely generated and torsion over R, if $\mu = 0$ then they vanish after localisations at height one singular primes.

Therefore the precise choice of $C^\bullet_S(\mathcal{L}/K)$ does not matter when $\mu = 0$, provided that it is perfect and has the correct cohomology.

However, without $\mu = 0$, the choice of $C^\bullet_S(\mathcal{L}/K)$ *does* matter!

How to prove the abelian EIMC without assuming $\mu = 0$

Recall L/K is a certain abelian CM extension such that $\zeta_p \in L$.
We assume $\mathrm{SBS}(L_n/L, S, T, p)$ for all n.
We want to prove the EIMC for L_∞^+/K.

Let $\mathcal{G} = \mathrm{Gal}(L_\infty/K)$ and let $\mathcal{G}^+ = \mathrm{Gal}(L_\infty^+/K)$.
Let $A_{L_n}^T = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{cl}_{L_n}^T$ and let $A_L^T = \varinjlim_n A_{L_n}^T$.
Let $\Psi_{S,T}$ be a T-modified version of a twist of Φ_S.

We show the existence of a $\Lambda(\mathcal{G})_-$-module $Y_S^T(-1)$ such that
(i) $Y_S^T(-1)$ is R-torsion,
(ii) $Y_S^T(-1)$ is of projective dimension 1 over $\Lambda(\mathcal{G})$, and
(iii) there is a surjection $Y_S^T(-1) \to \mathrm{Hom}(A_{L_\infty}^T, \mathbb{Q}_p/\mathbb{Z}_p)$.
Note that the construction of (iii) uses $C_S^\bullet(\mathcal{L}/K)$.

How to prove the abelian EIMC without assuming $\mu = 0$

We show that the EIMC for L_∞^+/K holds if and only if
$(\star)$ $\Psi_{S,T}$ is a generator of $\mathrm{Fitt}_{\Lambda(\mathcal{G})_-}(Y_S^T(-1))$.

An algebraic lemma shows that it suffices to show

(i) $(\star)$ after extending scalars to $\mathcal{M}(\mathcal{G})_-$ (this is the IMC)

(ii) $\Psi_{S,T}\Lambda(\mathcal{G})_\mathfrak{q} \subseteq \mathrm{Fitt}_{\Lambda(\mathcal{G})_-}(Y_S^T(-1))_\mathfrak{q}$ for every singular height one prime ideal $\mathfrak{q}$.

We prove that $Y_S^T(-1)_\mathfrak{q} \simeq \mathrm{Hom}(A_{L_\infty}^T, \mathbb{Q}_p/\mathbb{Z}_p)_\mathfrak{q}$.

Thus it suffices to show $\Psi_{S,T} \in \mathrm{Fitt}_{\Lambda(\mathcal{G})_-}(\mathrm{Hom}(A_{L_\infty}^T, \mathbb{Q}_p/\mathbb{Z}_p))$.

But $\Psi_{S,T} = \varprojlim_n \theta_S^T(L_n/K)^\#$ and
$\mathrm{Fitt}_\Lambda(\mathrm{Hom}(A_{L_\infty}^T, \mathbb{Q}_p/\mathbb{Z}_p)) = \varprojlim_n \mathrm{Fitt}_{\mathbb{Z}_p[G_n]_-}((A_{L_n}^T)^\vee)$.

So we are done by a lemma on Fitting ideals due to Greither and Kurihara since we know SBS$(L_n/L, S, T, p)$ for all n.

Proof of the EIMC when Sylow p-subgroup is abelian

Idea: show that certain products of maps are injective and exploit the functorial poperties of the conjecture to prove that the EIMC for $\mathcal{L}/K$ reduces to the EIMC for intermediate admissible extensions with p-elementary Galois group.

The Coates-Sinnott conjecture

Conjecture (Coates–Sinnott)

Let L/K be a finite abelian extension of number fields with Galois group G. Let r be a negative integer and let S be a finite set of places of K that contains all archimedean places and all places that ramify in L. Then one has

$$\mathrm{Ann}_{\mathbb{Z}[G]}(K_{1-2r}(\mathcal{O}_L)_{\mathrm{tors}})\theta_S(r) \subseteq \mathrm{Ann}_{\mathbb{Z}[G]}(K_{-2r}(\mathcal{O}_{L,S})).$$

Theorem (Nickel, J)

The Coates–Sinnott conjecture holds outside its 2-primary part.

Idea: reduce to CM-extensions and then show the plus / minus part of the p-part of the ETNC at $s = r$ depending on whether r is odd / even.