Hilbert modular forms

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Recall

- $\mathfrak{h}$: complex upper half plane
- F: totally real numberfield of degree m
- $GL_2^+(\mathbb{R})$ acts on $\mathfrak{h}$ by

$$\gamma \cdot z := \frac{az+b}{cz+d}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

- $GL_2^+(F)$ acts on $\mathfrak{h}^m$ by

$$GL_2^+(F) \hookrightarrow GL_2^+(\mathbb{R})^m$$

- $\Gamma \leq GL_2^+(F)$ is called congruence subgroup if $Z(GL_2^+(\mathcal{O}_F))\Gamma$ contains some finite index

$$\Gamma(\mathfrak{n}) = \{\gamma \in GL_2(\mathcal{O}_F) | \gamma \equiv \mathbb{1} \; mod \; \mathfrak{n}\}$$

Let $\gamma \in GL_2^+(\mathbb{R})$ and $z \in \mathfrak{h}$. Then define:

$$\mu(\gamma, z) := \frac{cz + d}{\sqrt{det(\gamma)}}$$

For $\gamma = (\gamma_1, \ldots, \gamma_m) \in GL_2^+(\mathbb{R})^m, z = (z_1, \ldots, z_m) \in \mathfrak{h}^m$ and $k = (k_1, \ldots, k_m) \in \mathbb{Z}^m$ define

$$\mu(\gamma, z)^k := \prod_{j=1}^{m} \mu(\gamma_j, z_j)^{k_j}$$

- Γ: congruence subgroup
- $k \in \mathbb{Z}^m$
- $\chi\colon \Gamma \to \mathbb{C}^\times$ with finite image
- $f\colon \mathfrak{h}^m \to \mathbb{C}$ holomorphic
- $\big(f|_k\gamma\big)(z) := f(\gamma \cdot z)\mu(\gamma, z)^{-k}$

Definition

f is called weak Hilbert modular form of weight k, level Γ and with character χ if

$$f|_k\gamma = \chi(\gamma)f \quad \forall\gamma \in \Gamma$$

f is called weak modular form (without character) if $\chi = \mathbb{1}$.

- $\mathit{Wfm}(\Gamma, k, \chi)$ space of all weak Hilbert modular forms of level Γ, weight k and with character χ
- $\mathit{Wfm}(\Gamma, k) = \mathit{Wfm}(\Gamma, k, \mathbb{1})$
- $\mathit{Wfm}(k) := \bigcup_{\Gamma} \mathit{Wfm}(\Gamma, k)$

Proposition

Let $f \in Wfm(\Gamma, k)$. Then f has a Fourier expansion

$$f(z) = \sum_{\xi \in \Lambda^*} c_\xi(f) exp(2\pi i Tr(\xi z))$$

that converges absolutely and uniformly.

- $\Lambda = \{u \in F | \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in \Gamma\}$
- $\Lambda^* = \{u \in F | Tr(u\Lambda) \subset \mathcal{O}_F\}$
- $Tr \colon \mathbb{C}^m \to \mathbb{C}$ linear extension of $Tr_{F/\mathbb{Q}} \colon F \to \mathbb{Q}$.

Proof of the Proposition

Let $z = x + iy$. View f for $y = const \Rightarrow$ Fourier expansions

$$f(x+iy) = \sum_{\xi \in \Lambda^*} c_\xi(y) exp(2\pi i Tr(\xi z))$$

f holomorphic $\Rightarrow$ Cauchy-Riemann equations

$$i\partial_{x_j} f = \partial_{y_j} f$$

$$\Rightarrow -2\pi\xi_j c_\xi(y) = \partial_{y_j} c_\xi(y)$$

$$\Rightarrow c_\xi(y) = c_\xi \quad \underbrace{exp(-2\pi Tr(\xi y))}_{\text{does not increase for increasing } y_i}$$

Definition

A weak Hilbert modular form f (of level Γ, weight k) is called Hilbert modular form (of level Γ, weight k) if for all $g \in GL_2^+(F)$

$$c_\xi(f|_k g) \neq 0$$

only can happen for $\xi = 0$ or ξ totally positive.

- $Mfm(\Gamma, k)$ space of Hilbert modular forms of level Γ and weight k
- $Mfm(k) = \bigcup_\Gamma Mfm(\Gamma, k)$

Definition

A Hilbert modular form f (of level Γ and weight k) is called cuspform (of level Γ and weight k) if

$$c_0(f|_k g) = 0$$

for all $g \in GL_2^+(F)$.

- $Cfm(\Gamma, k)$ space of cuspforms of level Γ and weight k
- $Cfm(k) = \bigcup_\Gamma Cfm(\Gamma, k)$

Proposition

If $f \in Wfm(\Gamma, k)$ *and* $g \in GL_2^+(F)$ *then*

$$f|_k g \in Wfm(g^{-1}\Gamma g, k).$$

The same holds for Hilbert modular forms and cuspforms.

Proof of the Proposition

$$(f|_k g)|_k(g^{-1}\gamma g) = (f|_k\gamma)|_k g = f|_k g$$

$$c_\xi(f|_k h) = 0 \forall h \in GL_2^+(F), \xi \not\geq 0$$

$$\Rightarrow c_\xi(f|_k gh) = 0 \forall h \in GL_2^+(F), \xi \not\geq 0$$

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$$\Rightarrow c_\xi(f|_k gh) = 0 \forall h \in GL_2^+(F), \xi \not> 0$$

Corollary

Let $f \in Wfm(\Gamma, k)$. *Then* f *is a cuspform if*

$$c_\xi(f|_k\delta_i) = 0 \;\forall i,\; \forall \xi \text{ not tot. pos.}$$

- P: subgroup of upper triangular matrices in $GL_2^+(F)$
- $\{\delta_i\}$: representatives of double coset $\Gamma \backslash GL_2^+(F)/P$

Proof of the Corollary

$g \in GL_2^+(F)$ can be written as product $g = \gamma\delta_i\rho$ with $\gamma \in \Gamma, \rho \in P$

$$f|_k g = \left(f|_k\delta_i\right)|_k\rho$$

So let f be such that $c_\xi(f) = 0$ for all ξ not totally positive and let $\rho = \begin{pmatrix} a & b \\ & d \end{pmatrix}$.

$$(f|_k\rho)(z) = f(\frac{az+b}{d})\mu(\rho, z)^{-k} = (\frac{a}{d})^{\frac{k}{2}} \sum_\xi c_\xi(f) exp(2\pi i Tr(\xi\frac{az+b}{d}))$$

$$= \sum_\xi (\frac{a}{d})^{\frac{k}{2}} c_{\xi\frac{d}{a}} exp(2\pi i Tr(\xi\frac{b}{a}))exp(2\pi i Tr(\xi z))$$

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$$= \sum_\xi \underbrace{(\frac{a}{d})^{\frac{k}{2}} c_{\xi\frac{d}{a}} exp(2\pi i Tr(\xi\frac{b}{a}))}_{c_\xi(f|_k\rho)} exp(2\pi i Tr(\xi z))$$

□

Theorem

The space of cuspforms is finite dimensional.

Theorem (Koecher's Principle)

Let $F \neq \mathbb{Q}$. And let f be a weak Hilbert modular form of level Γ and weight k.
(1) Then $c_\xi(f) \neq 0$ is only possible for ξ totally not negative.
(2) If $k \notin \mathbb{Z} \overset{diagonal}{\hookrightarrow} \mathbb{Z}^m$ then $c_0(f) = 0$.

Corollary

If $F \neq \mathbb{Q}$ each weak Hilbert modular form is a Hilbert modular form and if $k \notin \mathbb{Z}$ it is also a cuspform.

Proof of Koecher's Principle

$$\left\{ \tilde{y} = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} : y \in U \right\} \subset \Gamma$$

where $U \leq \mathcal{O}_F^\times$ has finite index
We get:

$$f\left(\eta^2 z\right) = \chi(\tilde{\eta})\eta^{-k} f(z)$$

Fourier coefficients:

$$c\left(\eta^{-2}\xi\right) = \chi(\tilde{\eta})\eta^{-k} c(\xi)$$

In particular:

$$c(0) = \chi(\tilde{\eta})\eta^{-k} c(0)$$

Assume $c(0) \neq 0$. There are $m-1$ independent $a_i \in ker(\chi)$ (viewed as character on U). We have:

$$a_i^{-k} = 1 \ \ \forall i$$

Hence

$$\sum_{j=1}^{m} log(\sigma_j(a_i))(-k_j) = 0 \ \ \forall i$$

So $-k$ is in the kernel of the matrix

$$(log(\sigma_j(a_i))_{i=1,...,m-1,j=1,...,m}$$

But this matrix has rank $m-1$, so the kernel is 1-dimensional and also $(1, ..., 1)$ is in the kernel, so $k \in \mathbb{Z}$.
This proves (2).

Assume there is some ξ not totally non-negative such that $c(\xi) \neq 0$ (say $\xi_j < 0$)
Take $\eta \in U$ so that $\sigma_j(\eta) >> 0$ and $0 < \sigma_i(\eta) < 1$ for $i \neq j$. Let $\lambda > 0$.
Subseries:

$$\sum_{n \geq 0} c\left(\eta^{2n}\xi\right) \exp\left(2\pi i \operatorname{Tr}\left(\eta^{2n}\xi i \lambda\right)\right)$$

$$= c(\xi) \sum_{n \geq 0} \chi(\tilde{\eta})^{-n} \eta^{nk} \exp\left(-2\pi\lambda \operatorname{Tr}\left(\eta^{2n}\xi\right)\right)$$

$$\operatorname{Tr}\left(\eta^{2n}\xi\right) = \eta_j^{2n}\xi_j + \sum_{i \neq j} \eta_i^{2n}\xi_i$$

cannot converge absolutely! This proves (1). □

Comparison to adelic viewpoint

- $\Gamma \leq SL_2(\mathbb{R})^m$ discrete
- $\kappa \in \mathbb{Z}^m$
- $f : \mathfrak{h}^m \to \mathbb{C}$ a function with
$$f|_\kappa \gamma = f$$
for every $\gamma \in \Gamma$
- $$f^\#(g) = f(g(i))\mu(g,i)^{-\kappa}$$
for $g \in SL(2,\mathbb{R})^m$
- $$k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2)^m$$
for $\theta \in \mathbb{R}^m$

Proposition

(1)

$$f^{\#}(\gamma g k(\theta)) = f^{\#}(g) e^{i\kappa\theta}$$

for $\gamma \in \Gamma$ *and* $\theta \in \mathbb{R}^m, g \in SL_2(\mathbb{R})^m$.
(2) Conversely, let φ *be function on* $SL(2,\mathbb{R})^m$ *so that* $\varphi(\gamma g k(\theta)) = \varphi(g) e^{i\kappa\theta}$ *for* $\gamma \in \Gamma$ *and* $\theta \in \mathbb{R}^m, g \in SL_2(\mathbb{R})^m$.

$$\varphi^{\natural}(x+iy) = y^{-\kappa/2} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\right)$$

is a function on $\mathfrak{h}^m$ *so that*

$$\varphi^{\natural}\Big|_{\mathrm{k}} \gamma = \varphi^{\natural}$$

for all $\gamma \in \Gamma$.
(3)

$$f = \left(f^{\#}\right)^{\natural}$$

Proof of the Proposition

(1)

$$f^{\#}(\gamma g k(\theta)) = f((\gamma g k(\theta)) \cdot i)\mu(\gamma g k(\theta), i)^{-\kappa}$$

$$= f((\gamma g) \cdot i)\underbrace{\mu(\gamma, (gk(\theta)) \cdot i)^{-\kappa}}_{\mu(\gamma, g \cdot i)^{-\kappa}}\mu(gk(\theta), i)^{-\kappa}$$

$$= (f|_{\kappa}\gamma)(g \cdot i)\mu(g, i)^{-\kappa}\mu(k(\theta), i)^{-\kappa} = f^{\#}(g)exp(\theta\kappa i)$$

Proof of the Proposition

(2) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $\theta = arg(cz+d)$. Then we have:

$$\gamma \overbrace{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}}^{g} k(\theta) = \cdots$$

$$= \begin{pmatrix} 1 & Re(\gamma \cdot z) \\ & 1 \end{pmatrix} \begin{pmatrix} Im(\gamma \cdot z)^{\frac{1}{2}} & \\ & Im(\gamma \cdot z)^{-\frac{1}{2}} \end{pmatrix} = (*)$$

$$(\varphi^{\natural}|_{\kappa}\gamma)(z) = y^{-\frac{\kappa}{2}} (\frac{1}{|cz+d|^2})^{-\frac{\kappa}{2}} \varphi((*))(cz+d)^{-\kappa}$$

$$= y^{-\frac{\kappa}{2}} \varphi(g) exp(i\kappa\theta) exp(i\theta)^{-\kappa} = \varphi^{\natural}(z)$$

Proof of the Proposition

(3)

$$(f^{\#})^{\natural}(x+iy) = y^{-\frac{\kappa}{2}} f^{\#}(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}}\end{pmatrix})$$

$$= y^{-\frac{\kappa}{2}} f(\underbrace{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}}\end{pmatrix}\cdot i}_{x+iy})\underbrace{\mu(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}}\end{pmatrix}, i)^{-\kappa}}_{y^{\frac{\kappa}{2}}}$$

$$= f(x+iy)$$

$\square$

- $\mathbb{A}$ adeles of F, $\mathbb{J}$ ideles of F
- $C_F^+ = \mathbb{J}/F_\infty^+ F^\times \hat{\mathcal{O}_F}^\times$
- $\hat{\mathcal{O}_F} := \prod_{\mathfrak{p}\nmid\infty} \mathcal{O}_{F_\mathfrak{p}}$
- $F_\infty := \prod_{\mathfrak{p}|\infty} F_\mathfrak{p} \cong \mathbb{R}^m$
- $\Gamma \leq SL_2(F)$ congruence subgroup if and only if

  $$\Gamma = SL_2(F) \cap SL_2(F_\infty)\mathbb{K}$$

  where $\mathbb{K} \leq SL_2(\hat{\mathcal{O}_F})$ compact open
- $\mathbb{K}(\mathfrak{n}) := \{\gamma \in SL_2(\hat{\mathcal{O}_F}) | \gamma \equiv \mathbb{1} \ mod \ \mathfrak{n}\}$
- $\tilde{\mathbb{K}}(\mathfrak{n}) := \{\gamma \in GL_2(\hat{\mathcal{O}_F}) | \gamma \equiv \mathbb{1} \ mod \ \mathfrak{n}\}$
- $\mathbb{K}_{00}(\mathfrak{n}) := \{\gamma \in SL_2(\hat{\mathcal{O}_F}) | \gamma \equiv \begin{pmatrix} * & \\ & * \end{pmatrix} \ mod \ \mathfrak{n}\}$
- $\tilde{\mathbb{K}}_{00}(\mathfrak{n}) := \{\gamma \in GL_2(\hat{\mathcal{O}_F}) | \gamma \equiv \begin{pmatrix} * & \\ & * \end{pmatrix} \ mod \ \mathfrak{n}\}$

Proposition

$$\iota \colon \Gamma g \to SL_2(F) g \mathbb{K}$$

gives rise to homeomorphism

$$\iota \colon \Gamma \backslash SL_2(F_\infty) \to SL_2(F) \backslash SL_2(\mathbb{A}) / \mathbb{K}$$

Proof of the Proposition

Well-defined: clear
Injective: components of γk at finite places are trivial
Surjective: Strong Approximation Theorem $\square$

Corollary

φ function on $SL_2(\mathbb{R})$ with

$$\varphi(\gamma g k(\theta)) = \varphi(g) exp(i\kappa\theta)$$

for all $\gamma \in \Gamma, \theta \in \mathbb{R}^m$. Define $\varphi^\#$ on $SL_2(\mathbb{A})$ by

$$\varphi^\#(\gamma g k) := \varphi(g)$$

for $\gamma \in SL_2(F), g \in SL_2(F_\infty), k \in \mathbb{K}$. Then this function is left-$SL_2(F)-$invariant, right $\mathbb{K}-$invariant and right $SO_2(\mathbb{R})^m-$equivariant. Any such function comes from a unique φ (restriction)

We say that $\varphi, \varphi^\#$ have weight κ.

Proposition

Assume $C_F^+ = 1$ *and let* $\mathfrak{n} \leq \mathcal{O}_F$ *be a proper ideal. Then* $\exists$ *natural homeomorphism:*

$$SL_2(F)\backslash SL_2(\mathbb{A})/\mathbb{K}_{00}(\mathfrak{n})\{\pm 1\}^m \to Z(F_\infty)GL_2(F)\backslash GL_2(\mathbb{A})/\tilde{\mathbb{K}}_{00}$$

Definition

$$M(\mathfrak{n},\kappa) := \{f: SL_2(\mathbb{A}) \to \mathbb{C} | SL_2(F) - l.i.\ ,\ weight\ \kappa\ ,\ \mathbb{K}(\mathfrak{n}) - r.i.\}$$

$\mathbb{K}_{00}(\mathfrak{n})$ acts on this space and

$$H := \mathbb{K}_{00}(\mathfrak{n})/\mathbb{K}(\mathfrak{n})$$

is finite.

$$M(\mathfrak{n},\kappa) = \bigoplus_{\chi} M(\mathfrak{n},\kappa,\chi)$$

Extend χ to $\tilde{\mathbb{K}}_{00}(\mathfrak{n})$ by setting

$$\chi(\begin{pmatrix} \lambda & \\ & 1 \end{pmatrix} k) := \chi(k), \lambda \in \hat{\mathcal{O}_F}^{\times}$$

Let $\tilde{\chi}$ be a (finite-order) unitary Hecke character with

$$(1) \quad \tilde{\chi}(\begin{pmatrix} t & \\ & t \end{pmatrix}) = \chi(\begin{pmatrix} t & \\ & t \end{pmatrix}), t \in \hat{\mathcal{O}_F}^{\times}$$

$$(2) \quad \tilde{\chi}(Z^{+}) = 1$$

$$(3) \quad \tilde{\chi}(\exp(i\theta)) = exp(iTr(\kappa\theta)), \theta \in \mathbb{Z}^{m}$$

Proposition

Assume $C_F^+ = 1$ *and let* $f \in M(\mathfrak{n}, \kappa, \chi)$*. Define* $f^\#$ *on* $GL_2(\mathbb{A})$ *by*

$$f^\#(\zeta\gamma g k) := f(g)\chi(k)$$

where $\gamma \in GL_2(F), g \in SL_2(\mathbb{A}), k \in \tilde{\mathbb{K}}_{00}(\mathfrak{n}), \zeta \in Z^+$*. This function is well-defined, of weight* κ *and*

$$f^\#(\begin{pmatrix} t & \\ & t \end{pmatrix}\gamma g k) = f(g)\chi(k)\tilde{\chi}(t)$$

Conversely for any such function there is a unique f *that gives rise to it.*

Now let $X \subset GL_2(\mathbb{A})$ be a set such that for each $\alpha \in C_F^+$ there is exactly one $\xi \in X$ such that

$$[det(\xi)] = \alpha$$

Lemma

$$GL_2(\mathbb{A}) = \dot{\bigcup}_{\xi \in X} Z(F_\infty) GL_2(F) SL_2(\mathbb{A}) \xi \tilde{\mathbb{K}}_{00}(\mathfrak{n})$$

$$\dot{\bigcup}_{\xi \in X} F_\infty^+ F^\times \hat{\mathcal{O}}_F^{\times} det(\xi) = \mathbb{J}$$

□

Lemma

Let $g, g' \in SL_2(\mathbb{A})$. *Then*

$$g' \in Z(F_\infty) GL_2(F) g \tilde{\mathbb{K}}_{00}(\mathfrak{n})$$

$$\Leftrightarrow g' \in \begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix} SL_2(F) g \mathbb{K}_{00}(\mathfrak{n}) \{\pm 1\}^m$$

for some $y \in \mathcal{O}_F^\times$ *totally positive.*

$$\Gamma_\xi := GL_2^+(F) \cap GL_2^+(F_\infty)\xi\tilde{\mathbb{K}}_{00}(\mathfrak{n})\xi^{-1}$$

Proposition

There is a homeomorphism

$$\underbrace{Z(F_\infty)GL_2(F)\backslash GL_2(\mathbb{A})/\tilde{\mathbb{K}}_{00}(\mathfrak{n})}_{\mathcal{G}} \cong \dot{\bigcup}_{\xi \in X} \underbrace{Z(F_\infty)\Gamma_\xi \backslash GL_2^+(F_\infty)}_{\Phi_\xi}$$

Proof of the Proposition

Define

$$\iota \colon \dot{\bigcup}_{\xi \in X} \Phi_\xi \to \mathcal{G}$$

by

$$\overline{g} \mapsto \overline{g\xi}, \quad \overline{g} \in X_\xi$$

Definition of Γ_ξ, determinants + Strong Approximation $\Rightarrow$ well-defined and surjective

Injectivity: If $g, h \in GL_2(F_\infty)$ with $\iota(\overline{g}) = \iota(\overline{h})$ then

$$h\xi = \zeta\gamma g\xi k$$

where $\zeta \in Z(F_\infty), \gamma \in GL_2(F), k \in \tilde{\mathbb{K}}_{00}(\mathfrak{n})$.

$$\Rightarrow \gamma \in \Gamma_\xi$$

Corollary

$$Z(F_\infty)GL_2(F)\backslash GL_2(\mathbb{A})/\tilde{\mathbb{K}}_{00}(\mathfrak{n})SO(2)^m$$

is a finite union of quotients of $\mathfrak{h}^m$ *by congruence subgroups.*