Group ring valued Hilbert modular forms and Eisenstein series

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Contents

Contents

- $\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ upper half plane
- $k > 2$ even integer
- $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$ acts on $f : \mathcal{H} \to \mathbb{C}$ by

$$f \mid_{k,\gamma} (z) = \det(\gamma)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

Aim

Construct an f that is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$.

Idea: Let G be a group acting on some G-module V. Let $v_0 \in V$.
If G is finite, then $\sum_{g \in G} g(v_0) \in V^G$.
If G is infinite, but the stabilizer G_0 of v_0 in G has finite index, then $\sum_{g \in G/G_0} g(v_0) \in V^G$.

A first Eisenstein series

Consider $\mathbf{1} : \mathcal{H} \to \mathbb{C}$, $z \mapsto 1$ and let $\gamma \in \Gamma := \mathrm{SL}_2(\mathbb{Z})$. Then

$$\mathbf{1} \mid_{k,\gamma} (z) = (cz+d)^{-k} = 1 \iff c = 0 \,(\text{hence } d = \pm 1)$$

$$\iff \gamma \in \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} = \Gamma_\infty .$$

Define

$$\tilde{E}_k(z) := \sum_{\Gamma_\infty \backslash \Gamma \ni \gamma} \mathbf{1} \mid_{k,\gamma} (z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} (cz+d)^{-k}$$

is absolutely convergent and a modular form of weight k.

Different normalizations

$$\begin{aligned} G_k(z) &:= \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} (mz+n)^{-k} \\ &= \zeta(k)\tilde{E}_k(z) = \dots \\ &= \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1} q^{mr}, \quad q = \exp(2\pi i z) \\ &= \frac{(2\pi i)^k}{(k-1)!} \left(\frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \right) =: \frac{(2\pi i)^k}{(k-1)!} E_k(z) \end{aligned}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.
Note that $E_k(z)$ has rational coefficients!

We use Euler's identity

$$\frac{\pi}{\tan(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{z+n}.$$

The function on the left has Fourier expansion

$$\begin{aligned} \frac{\pi}{\tan(\pi z)} &= \pi\frac{\cos(\pi z)}{\sin(\pi z)} = \pi i \frac{\exp(\pi i z)+\exp(-\pi i z)}{\exp(\pi i z)-\exp(-\pi i z)} \\ &= -\pi i \frac{1+q}{1-q} = -2\pi i\left(\frac{1}{2}+\sum_{r=1}^{\infty} q^r\right). \end{aligned}$$

We differentiate $k-1$ times and obtain

$$\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^k} = \frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{r=1}^{\infty} r^{k-1}q^{mr} = \frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

- $N > 2$ integer
- $\Gamma_0(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N\right\}$
- $\Gamma_1(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N\right\}$
- $\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d$ induces

$$\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$$

- $\mathcal{M}_k(\Gamma_1(N)) \simeq \bigoplus_{\substack{\chi:(\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times \\ \chi(-1)=(-1)^k}} \mathcal{M}_k(\Gamma_0(N), \chi)$.
- $f \in \mathcal{M}_k(\Gamma_0(N), \chi)$ means $f \mid_{k,\gamma} (z) = \chi(d) f(z)$ for all $\gamma \in \Gamma_0(N)$.
- in particular $(-1)^{-k} f(z) = f \mid_{k,-\mathbf{1}} (z) = \chi(-1) f(z)$.

More Eisenstein series

Define

$$\begin{aligned}\tilde{E}_k(1,\chi) &:= \sum_{\Gamma_\infty\backslash\Gamma_0(N)\ni\gamma} \chi(d)(cz+d)^{-k} \in \mathcal{M}_k(\Gamma_0(N),\chi)\\ &= 1+\frac{2}{L_N(1-k,\chi)}\sum_{n=1}^{\infty}\left(\sum_{d|n}\chi(d)d^{k-1}\right)q^n.\end{aligned}$$

More generally, for $\psi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^\times$ one has

$$\begin{aligned}E_k(\psi,\chi) &:= \delta_{\psi,1}\frac{L_N(1-k,\chi)}{2}+\sum_{n=1}^{\infty}\left(\sum_{d|n}\psi\left(\frac{n}{d}\right)\chi(d)d^{k-1}\right)q^n\\ &\in \mathcal{M}_k(\Gamma_0(MN),\psi\chi)\end{aligned}$$

Hecke operators

- Γ_1, Γ_2 congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$
- $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$
- Consider the double coset $\Gamma_1 \alpha \Gamma_2 = \dot{\bigcup}_j \Gamma_1 \beta_j$ and define

$$\begin{aligned} [\Gamma_1 \alpha \Gamma_2]_k : \mathcal{M}_k(\Gamma_1) &\to \mathcal{M}_k(\Gamma_2) \\ f &\mapsto \sum_j f \mid_{k,\beta_j} \end{aligned}$$

Example ($\Gamma_1 \supset \Gamma_2$, $\alpha = 1$)

$$\mathcal{M}_k(\Gamma_1) \hookrightarrow \mathcal{M}_k(\Gamma_2)$$

Example ($\Gamma_1 \subset \Gamma_2$, $\alpha = 1$)

$$\mathrm{Tr}_{\Gamma_2/\Gamma_1} : \mathcal{M}_k(\Gamma_1) \to \mathcal{M}_k(\Gamma_2)$$

Hecke operators II

Example (diamond operator: $\Gamma_1 = \Gamma_2 = \Gamma_1(N)$, $\alpha \in \Gamma_0(N)$)

$$\langle d \rangle : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N)),$$

where $\alpha \mapsto d \mod N$ under $\Gamma_0(N)/\Gamma_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$.

Since $\Gamma_1(N)$ is normal in $\Gamma_0(N)$, we have

$$\Gamma_1(N)\alpha\Gamma_1(N) = \Gamma_1(N)\alpha.$$

Hence $\langle d \rangle = |_{k,\alpha}$ and

$$\mathcal{M}_k(\Gamma_0(N), \chi) = \{f \in \mathcal{M}_k(\Gamma_1(N)) \mid \langle d \rangle f = \chi(d) f\}.$$

Hecke operators III

Example ($\Gamma_1 = \Gamma_2 = \Gamma_1(N)$, $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$, p prime)

$$T_p : \mathcal{M}_k(\Gamma_1(N)) \to \mathcal{M}_k(\Gamma_1(N))$$

Remark: In the paper the notation T_p is reserved for primes $p \nmid N$. If $p \mid N$ the corresponding Hecke operator is denoted U_p.

Proposition

1. *If $f \in \mathcal{M}_k(\Gamma_0(N), \chi)$, so is $T_p f$.*
2. *If $f = \sum_{n=0}^{\infty} a_n q^n$ then*

$$T_p f = \sum_{n=0}^{\infty} (a_{np} + \chi(p) p^{k-1} a_{n/p}) q^n,$$

where $a_{n/p} = 0$ if $n/p \notin \mathbb{Z}$.

Contents

- F totally real field; $\quad n := [F : \mathbb{Q}]$
- $\mathrm{GL}_2^+(F) \hookrightarrow \mathrm{GL}_2^+(\mathbb{R})^n$
- $\mathfrak{d}$ different of F
- $\mathrm{Cl}^+(F) =$ $\{\text{fractional ideals}\} / \{(\alpha) \mid \alpha \in F^\times \text{ totally positive}\}$ narrow class group
- For each $\lambda \in \mathrm{Cl}^+(F)$ choose a representative ideal t_λ
- $\mathfrak{n} \subseteq \mathcal{O}_F$ ideal
- $\Gamma_\lambda(\mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F) \mid {a,d\in\mathcal{O}_F,\, c\in t_\lambda\mathfrak{d}\mathfrak{n},\, b\in(t_\lambda\mathfrak{d})^{-1}, \atop ad-bc\in\mathcal{O}_F^\times,\, d\equiv 1 \mod \mathfrak{n}} \right\}$
- This corresponds to $\Gamma_1(N)$ if $F = \mathbb{Q}$.

- k positive integer
- $\mathcal{M}_k(\mathfrak{n})$ Hilbert modular forms of level $\mathfrak{n}$ and weight k
- $f \in \mathcal{M}_k(\mathfrak{n})$ is a tuple

$$f = (f_\lambda)_{\lambda \in \mathrm{Cl}^+(F)}$$

of holomorphic functions

$$f_\lambda : \mathcal{H}^n \to \mathbb{C}$$

$$f_\lambda \mid_{k,\alpha} = f_\lambda \quad \forall \lambda \in \mathrm{Cl}^+(F) \, \forall \alpha \in \Gamma_\lambda(\mathfrak{n})$$

- Nils: f corresponds to a single map

$$\phi_f : Z(F_\infty)\mathrm{GL}_2(F) \backslash \mathrm{GL}_2(\mathbb{A}_F) \to \mathbb{C}$$

that is invariant under

$$\mathbb{K}(\mathfrak{n}) = \left\{ \gamma \in \mathrm{GL}_2(\hat{\mathcal{O}}_F) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod \mathfrak{n} \right\}$$

Cusp classes

Antonio:

$$\mathrm{cusps}(\Gamma_\lambda(\mathfrak{n})) := \Gamma_\lambda(\mathfrak{n})\backslash\mathbb{P}^1(F) \leftrightarrow \Gamma_\lambda(\mathfrak{n})\backslash\mathrm{GL}_2^+(F)/\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$\alpha(\infty) = [a : c] \hookleftarrow \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a finite set.

$$\mathrm{cusps}(\mathfrak{n}) := \coprod_\lambda \mathrm{cusps}(\Gamma_\lambda(\mathfrak{n}))$$

Each pair $\mathcal{A} = (A, \lambda) \in \mathrm{GL}_2^+(F) \times \mathrm{Cl}^+(F)$ gives rise to a class $[\mathcal{A}] \in \mathrm{cusps}(\mathfrak{n})$.

Normalized Fourier coefficients

Let $\mathcal{A} = (A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda) \in \mathrm{GL}_2^+(F) \times \mathrm{Cl}^+(F)$ and expand

$$f_\lambda \mid_{k,A} (z) = a_{\mathcal{A}}(0) + \sum_{\substack{\xi \in \Lambda \\ \xi \gg 0}} a_{\mathcal{A}}(\xi) \exp(2\pi i \mathrm{Tr}(\xi z))$$

for some lattice Λ (actually $\Lambda = t_\lambda$ if $A = 1$).

$$\begin{aligned} \mathfrak{b}_{\mathcal{A}} &:= (a) + c(\mathfrak{t}_\lambda \mathfrak{d})^{-1} \\ c_{\mathcal{A}}(\xi, f) &:= a_{\mathcal{A}}(\xi) \cdot N(\mathfrak{t}_\lambda)^{-k/2} \cdot N(\mathfrak{b}_{\mathcal{A}})^{-k} \cdot N(\det(A))^{k/2} \end{aligned}$$

Then $c_{\mathcal{A}}(0, f)$ only depends on the class $[\mathcal{A}]$ (up to sign): For $\gamma \in \Gamma_\lambda(\mathfrak{n})$, $A' = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix}$ one has

$$c_{(\gamma A A', \lambda)}(0, f) = \mathrm{sgn}(N(a'))^k c_{(A,\lambda)}(0, f).$$

q-expansion

$$c_\lambda(0,f) := c_{(1,\lambda)}(0,f) = a_{(1,\lambda)}(0)\cdot N(\mathfrak{t}_\lambda)^{-k/2}$$

Each $0 \neq \mathfrak{m} \subseteq \mathcal{O}_F$ can be written as $\mathfrak{m} = (\xi)\mathfrak{t}_\lambda^{-1}$ for some $\xi \in \mathfrak{t}_\lambda$, $\xi \gg 0$ and a unique $\lambda \in \mathrm{Cl}^+(F)$.

$$c(\mathfrak{m},f) := a_{(1,\lambda)}(\xi)\cdot N(\mathfrak{t}_\lambda)^{-k/2}$$

The collection $\{c_\lambda(0,f), c(\mathfrak{m},f)\}$ of normalized Fourier coefficients is called the q-expansion of f and determines f uniquely.

Cusps above infinity and zero

Let $\mathcal{A} = (A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \lambda) \in \mathrm{GL}_2^+(F) \times \mathrm{Cl}^+(F)$ and define

$$\mathfrak{c}_{\mathcal{A}} := (c)(t_\lambda \mathfrak{d} \mathfrak{b}_{\mathcal{A}})^{-1} \subseteq \mathcal{O}_F.$$

For $\mathfrak{b} \mid \mathfrak{n}$ we set

$$\begin{aligned} \infty \in C_\infty(\mathfrak{b}, \mathfrak{n}) &:= \{[\mathcal{A}] \in \mathrm{cusps}(\mathfrak{n}) : \mathfrak{b} \mid \mathfrak{c}_{\mathcal{A}}\} \\ 0 \in C_0(\mathfrak{b}, \mathfrak{n}) &:= \{[\mathcal{A}] \in \mathrm{cusps}(\mathfrak{n}) : (\mathfrak{b}, \mathfrak{c}_{\mathcal{A}}) = 1\} \end{aligned}$$

as $\mathfrak{c}_\infty = 0$ and $\mathfrak{c}_0 = \mathcal{O}_F$ for each $0, \infty \in \Gamma_\lambda(\mathfrak{n}) \backslash \mathbb{P}^1(F)$.

$$\begin{aligned} C_\infty(\mathfrak{n}) &:= C_\infty(\mathfrak{n}, \mathfrak{n}) \\ C_0(\mathfrak{n}) &:= C_0(\mathfrak{n}, \mathfrak{n}) \end{aligned}$$

Hilbert modular forms with nebentypus

- $G_{\mathfrak{n}}^{+}$ narrow ray class group of conductor $\mathfrak{n}$
- A class $\mathfrak{a} \in G_{\mathfrak{n}}^{+}$ acts upon $\mathcal{M}_k(\mathfrak{n})$ via a diamond operator $S(\mathfrak{a})$ so that

$$\mathcal{M}_k(\mathfrak{n}) = \bigoplus_{\psi} \mathcal{M}_k(\mathfrak{n}, \psi),$$

where $f \in \mathcal{M}_k(\mathfrak{n}, \psi)$ iff $f \in \mathcal{M}_k(\mathfrak{n})$ and $S(\mathfrak{a})f = \psi(\mathfrak{a})f$ for all $\mathfrak{a} \in G_{\mathfrak{n}}^{+}$.
(Hilbert modular forms with nebentypus ψ)

- Here, the sum runs over all characters $\psi : G_{\mathfrak{n}}^{+} \to \mathbb{C}^{\times}$ such that 'ψ has sign $(k, \ldots, k)$':

$$\psi((\alpha)) = \mathrm{sgn}(N(\alpha))^k \quad \forall \alpha \equiv 1 \mod \mathfrak{n}.$$

What is the diamond operator?

Consider functions

$$\phi : Z(F_\infty)\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/\mathbb{K}(\mathfrak{n}) \to \mathbb{C}$$

Then $a \in \mathbb{A}_F^\times = Z(\mathrm{GL}_2(\mathbb{A}_F))$ acts on ϕ via $S(a)\phi(x) := \phi(ax)$. Since $F^\times$ acts trivially, this factors through an action of the idèle class group C_F and indeed through an action of $G_\mathfrak{n}^+$.
If $a \in \mathbb{A}_F^\times$ maps to $\mathfrak{a} \in G_\mathfrak{n}^+$, then $S(\mathfrak{a})f$ corresponds to $S(a)\phi_f$.

Level raising

Proposition (Shimura)

Let $f \in \mathcal{M}_k(\mathfrak{n})$ *and let* $\mathfrak{q} \subseteq \mathcal{O}_F$.
Then there is a form $f\mid_{\mathfrak{q}} \in \mathcal{M}_k(\mathfrak{n}\mathfrak{q})$ *such that*

$$c(\mathfrak{m}, f\mid_{\mathfrak{q}}) = \begin{cases} c(\mathfrak{m}/\mathfrak{q}, f) & \textit{if} \quad \mathfrak{q} \mid \mathfrak{m} \\ 0 & \textit{if} \quad \mathfrak{q} \nmid \mathfrak{m} \end{cases}$$

and

$$c_\lambda(0, f\mid_{\mathfrak{q}}) = c_{\lambda\mathfrak{q}}(0, f).$$

Contents

Let A be a commutative ring and $\psi : G_{\mathfrak{n}}^{+} \to A^{\times}$ a character.

$$\begin{aligned} \mathcal{M}_k(\mathfrak{n}, \mathbb{Z}) &:= \{f \in \mathcal{M}_k(\mathfrak{n}) \mid c(\mathfrak{m}, f), c_\lambda(0, f) \in \mathbb{Z}\} \\ \mathcal{M}_k(\mathfrak{n}, A) &:= \mathcal{M}_k(\mathfrak{n}, \mathbb{Z}) \otimes A \\ \mathcal{M}_k(\mathfrak{n}, A, \psi) &:= \left\{f \in \mathcal{M}_k(\mathfrak{n}, A) \mid S(\mathfrak{a})f = \psi(\mathfrak{a})f \; \forall \mathfrak{a} \in G_{\mathfrak{n}}^{+}\right\} \end{aligned}$$

We define $\mathcal{S}_k(\mathfrak{n}, A)$ and $\mathcal{S}_k(\mathfrak{n}, A, \psi)$ similarly.
In particular, consider

$$\boldsymbol{\psi} : G_{\mathfrak{n}}^{+} \twoheadrightarrow G \to R_{\Psi}^{\times},$$

where $R = R_{\Psi}$ is a character group ring, and the space of group ring valued Hilbert modular forms

$$\mathcal{M}_k(\mathfrak{n}, R, \boldsymbol{\psi}).$$

Recall that $R \hookrightarrow \prod_{\psi \in \Psi} \mathcal{O}$.

Lemma (essentially due to Silliman)

Let $c(\mathfrak{m}), c_\lambda(0) \in R$ such that for all $\psi \in \Psi$ there is a form $f_\psi \in \mathcal{M}_k(\mathfrak{n}, \mathcal{O}, \psi)$ with $c(\mathfrak{m}, f_\psi) = \psi(c(\mathfrak{m}))$ and $c_\lambda(0, f_\psi) = \psi(c_\lambda(0))$. Then there exists a unique $f \in \mathcal{M}_k(\mathfrak{n}, R, \psi)$ such that $\psi(f) = f_\psi$ for all $\psi \in \Psi$.

Proof.

Silliman:

$$\mathcal{M}_k(\mathfrak{n}, R) = \{f \in \prod_{\psi \in \Psi} \mathcal{M}_k(\mathfrak{n}, \mathcal{O}) \mid c(\mathfrak{m}, f), c_\lambda(0, f) \in R\}.$$

Hence there is a unique $f \in \mathcal{M}_k(\mathfrak{n}, R)$ such that $\psi(f) = f_\psi$ for all $\psi \in \Psi$.

$f_\psi \in \mathcal{M}_k(\mathfrak{n}, \mathcal{O}, \psi) \implies f \in \mathcal{M}_k(\mathfrak{n}, R, \psi)$. □

Hecke operators

Let $\mathfrak{q}$ be a prime such that $\mathfrak{q} \nmid \mathfrak{n}$.
Let $f \in \mathcal{M}_k(\mathfrak{n}, R, \psi)$. Then there is $T_\mathfrak{q} f \in \mathcal{M}_k(\mathfrak{n}, R, \psi)$ such that

$$\begin{aligned} c(\mathfrak{m}, T_\mathfrak{q} f) &= \sum_{\mathfrak{a} \mid (\mathfrak{m}, \mathfrak{q})} \psi(\mathfrak{a}) N(\mathfrak{a})^{k-1} c(\mathfrak{m}\mathfrak{q}/\mathfrak{a}^2, f) \\ c_\lambda(0, T_\mathfrak{q} f) &= c_{\lambda\mathfrak{q}^{-1}}(0, f) + \psi(\mathfrak{q}) N(\mathfrak{q})^{k-1} c_{\lambda\mathfrak{q}}(0, f). \end{aligned}$$

Same formulae for $U_\mathfrak{q}$ if $\mathfrak{q} \mid \mathfrak{n}$ with the convention that $\psi(\mathfrak{q}) = 0$.
One has

$$U_\mathfrak{q}(f \mid_\mathfrak{q}) = f \in \mathcal{M}_k(\mathfrak{n}\mathfrak{q}, R, \psi).$$

Let $\psi : G_{\mathfrak{n}}^{+} \to \mathcal{O}^{\times}$, $\chi : G_{\mathfrak{n}'}^{+} \to \mathcal{O}^{\times}$ be characters such that $\psi\chi, \psi\chi^{-1} : G_{\mathfrak{n}\mathfrak{n}'}^{+} \to \mathcal{O}^{\times}$ have sign $(k, \ldots, k)$, $k \geq 1$.

Proposition (Existence of Eisenstein series)

There is a Hilbert modular form $E_k(\psi, \chi) \in \mathcal{M}_k(\mathfrak{n}\mathfrak{n}', \psi\chi)$ *such that*

$$c(\mathfrak{m}, E_k(\psi, \chi)) = \sum_{\mathfrak{r}|\mathfrak{m}} \psi\left(\frac{\mathfrak{m}}{\mathfrak{r}}\right) \chi(\mathfrak{r}) N(\mathfrak{r})^{k-1}.$$

If $k > 1$ *then*

$$c_\lambda(0, E_k(\psi, \chi)) = 2^{-n} \cdot \begin{cases} \psi^{-1}(\lambda) L(\psi\chi^{-1}, 1-k) & \textit{if} \quad \mathfrak{n} = 1 \\ 0 & \textit{if} \quad \mathfrak{n} \neq 1 \end{cases}$$

Let $\psi : G_{\mathfrak{n}}^{+} \to \mathcal{O}^{\times}$, $\chi : G_{\mathfrak{n}'}^{+} \to \mathcal{O}^{\times}$ be characters such that $\psi\chi, \psi\chi^{-1} : G_{\mathfrak{n}\mathfrak{n}'}^{+} \to \mathcal{O}^{\times}$ have sign $(k, \dots, k)$, $k \geq 1$.

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$$c(\mathfrak{m}, E_k(\psi, \chi)) = \sum_{\mathfrak{r}|\mathfrak{m}} \psi\left(\frac{\mathfrak{m}}{\mathfrak{r}}\right) \chi(\mathfrak{r}) N(\mathfrak{r})^{k-1}.$$

If $k = 1$ then $E_1(\psi, \chi) = E_1(\chi, \psi)$ and

$$c_\lambda(0, E_k(\psi, \chi)) = 2^{-n} \cdot \begin{cases} \psi^{-1}(\lambda) L(\chi\psi^{-1}, 0) & \text{if} \quad \mathfrak{n} = 1 \neq \mathfrak{n}' \\ \chi^{-1}(\lambda) L(\psi\chi^{-1}, 0) & \text{if} \quad \mathfrak{n}' = 1 \neq \mathfrak{n} \\ \substack{\psi^{-1}(\lambda) L(\chi\psi^{-1}, 0) \\ +\chi^{-1}(\lambda) L(\psi\chi^{-1}, 0)} & \text{if} \quad \mathfrak{n} = \mathfrak{n}' = 1 \\ 0 & \text{if} \quad \mathfrak{n} \neq 1 \neq \mathfrak{n}' \end{cases}$$

- H/F abelian CM extension with Galois group G
- $\psi : G \to \mathcal{O}^\times$ totally odd character
- S finite set of places of F such that $\mathfrak{p} \in S$ for all $\mathfrak{p} \mid \mathrm{cond}(\psi)$
- $\mathfrak{n} := \mathrm{cond}(\psi) \prod_{\mathfrak{p} \in S, \mathfrak{p} \nmid \mathrm{cond}(\psi)} \mathfrak{p}$
- ψ induces a character $\psi_S : G_{\mathfrak{n}}^+ \to \mathcal{O}^\times$
- $E_k(\psi_S, 1) \in \mathcal{M}_k(\mathfrak{n}, \mathcal{O}, \psi)$ is an eigenvector for the Hecke operators:

$$\begin{aligned} T_{\mathfrak{q}} E_k(\psi_S, 1) &= (\psi(\mathfrak{q}) + N(\mathfrak{q})^{k-1}) \cdot E_k(\psi_S, 1) \\ U_{\mathfrak{q}} E_k(\psi_S, 1) &= N(\mathfrak{q})^{k-1} \cdot E_k(\psi_S, 1) \end{aligned}$$

If $R = R_\Psi$ is a character group ring with canonical character $\boldsymbol{\psi} : G_{\mathfrak{n}}^{+} \to R^{\times}$, then these Eisenstein series almost fit into a group ring family:
There is $E_k(\boldsymbol{\psi}, 1) \in \mathcal{M}_k(\mathfrak{n}, \mathrm{Frac}(R), \boldsymbol{\psi})$ with specialization at $\psi \in \Psi$ equal to $E_k(\psi_S, 1)$:

$$c(\mathfrak{m}, E_k(\boldsymbol{\psi}, 1)) = \sum_{\substack{\mathfrak{r}|\mathfrak{m} \\ (\mathfrak{m}/\mathfrak{r}, \mathfrak{n})=1}} \boldsymbol{\psi}\left(\frac{\mathfrak{m}}{\mathfrak{n}}\right) N(\mathfrak{r})^{k-1} \in R,$$

but

$$c_\lambda(0, E_k(\boldsymbol{\psi}, 1)) = 2^{-n} \cdot \begin{cases} 0 & \text{if} \quad k > 1, \mathfrak{n} \neq 1 \\ \boldsymbol{\psi}^{-1}(\lambda)\Theta_S(1-k) & \text{if} \quad k > 1, \mathfrak{n} = 1 \\ \Theta_S^{\sharp}(0) & \text{if} \quad k = 1, \mathfrak{n} \neq 1 \\ \Theta_S^{\sharp}(0) + \boldsymbol{\psi}(\lambda)\Theta_S(0) & \text{if} \quad k = 1, \mathfrak{n} = 1 \end{cases}$$

is only in $\mathrm{Frac}(R)$.

Ordinary forms

For $\mathfrak{p} \mid p$ and more generally for $\mathfrak{P}$ a product of such primes we set

$$e_{\mathfrak{p}}^{\mathrm{ord}} := \lim_{n\to\infty} U_{\mathfrak{p}}^{n!}, \quad e_{\mathfrak{P}}^{\mathrm{ord}} := \prod_{\mathfrak{p}\mid\mathfrak{P}} e_{\mathfrak{p}}^{\mathrm{ord}}$$

and define spaces of *p-ordinary forms*

$$\mathcal{M}_k(\mathfrak{n}, R, \psi)^{\mathfrak{P}-\mathrm{ord}} := e_{\mathfrak{P}}^{\mathrm{ord}} \mathcal{M}_k(\mathfrak{n}, R, \psi)$$
$$\mathcal{S}_k(\mathfrak{n}, R, \psi)^{\mathfrak{P}-\mathrm{ord}} := e_{\mathfrak{P}}^{\mathrm{ord}} \mathcal{S}_k(\mathfrak{n}, R, \psi).$$

$U_{\mathfrak{p}}$ acts invertibly on these spaces for each $\mathfrak{p} \mid \mathfrak{P}$.

We need to prove the following.

Lemma

Let $\mathcal{O}$ be the ring of integers in some finite extension K of $\mathbb{Q}_p$. Let A be a commutative $\mathcal{O}$-algebra of finite rank over $\mathcal{O}$.
Then for any $x \in A$ the limit $\lim_{n\to\infty} x^{n!}$ exists in A and is an idempotent in A.

Step 1: $A = \mathcal{O}$

Let $\mathfrak{p}$ be the maximal ideal in $\mathcal{O}$.
If $x \in \mathfrak{p}$, then clearly $\lim_{n\to\infty} x^{n!} = 0$.
So let $x \in \mathcal{O}^\times$. For $r > 0$ we have that

$$\sharp\left((\mathcal{O}/\mathfrak{p}^r)^\times\right) = p^{fr-1}(p^f - 1).$$

Hence $x^{p^{fr}(p^f-1)} \equiv 1 \mod \mathfrak{p}^r$ and thus

$$\lim_{n\to\infty} x^{n!} = \lim_{r\to\infty} x^{p^{fr}(p^f-1)} = 1.$$

Step 2: $A \otimes_{\mathcal{O}} K$ is semi-simple

We have

$$A \otimes_{\mathcal{O}} K \simeq \prod_{i=1}^{t} K_i,$$

where each K_i is finite over $\mathbb{Q}_p$.
Let $(x_i)_i$ be the image of $x \otimes 1$ under this isomorphism.
Then $x_i \in \mathcal{O}_{K_i}$ for each i and we can apply step 1 in each component.

Step 3: General A

Wedderburn's principal theorem: We may write

$$x = s + t \in A \otimes_{\mathcal{O}} K,$$

where t is nilpotent and s is semi-simple, i.e. the subalgebra $K[s]$ is semi-simple.
If $t^j = 0$, then

$$(s+t)^{p^{fr}} = s^{p^{fr}} + \sum_{i=1}^{j-1} \binom{p^{fr}}{i} s^{p^{fr}-i} t^i.$$

By elementary means we have $\left|\binom{p^{fr}}{i}\right|_p < Cp^{-fr}$ for a constant independent of r.

$$\lim_{n\to\infty} x^{n!} = \lim_{n\to\infty} s^{n!}$$

is an idempotent by step 2.

An example

Let $\psi : G_{\mathfrak{n}}^{+} \to \mathcal{O}^{\times}$ be a totally odd character. What is

$$e_{\mathfrak{p}}^{\mathrm{ord}} E_k(\psi, 1)$$

for $k > 1$ and $\mathfrak{p} \nmid \mathfrak{n}$?
Suppose f is a Hilbert modular form and an eigenform for $U_{\mathfrak{p}}$ with eigenvalue a. Then

$$e_{\mathfrak{p}}^{\mathrm{ord}} f = \begin{cases} f & \text{if} \quad |a|_p = 1 \\ 0 & \text{if} \quad |a|_p < 1. \end{cases}$$

We write

$$E_k(\psi,1) = E_k(\psi,1_{\mathfrak{p}}) + N(\mathfrak{p})^{k-1} E_k(\psi,1) \mid_{\mathfrak{p}} .$$

We have already observed that

$$U_{\mathfrak{p}} E_k(\psi,1) \mid_{\mathfrak{p}} = E_k(\psi,1).$$

$$U_{\mathfrak{p}} E_k(\psi,1_{\mathfrak{p}}) = \psi(\mathfrak{p}) E_k(\psi,1_{\mathfrak{p}}) \implies e_{\mathfrak{p}}^{\mathrm{ord}} E_k(\psi,1_{\mathfrak{p}}) = E_k(\psi,1_{\mathfrak{p}}).$$

Hence

$$(U_{\mathfrak{p}} - N(\mathfrak{p})^{k-1}) e_{\mathfrak{p}}^{\mathrm{ord}} E_k(\psi,1) = \psi(\mathfrak{p}) E_k(\psi,1_{\mathfrak{p}}).$$

Since $U_{\mathfrak{p}} - N(\mathfrak{p})^{k-1}$ is invertible on the ordinary part, we get

$$e_{\mathfrak{p}}^{\mathrm{ord}} E_k(\psi,1) = \frac{\psi(\mathfrak{p})}{\psi(\mathfrak{p}) - N(\mathfrak{p})^{k-1}} E_k(\psi,1_{\mathfrak{p}}).$$

Thank you for your attention!