The Brumer-Stark Conjecture - Talk 9

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Eisenstein series

Let $\chi : G_{\mathfrak{n}}^{+} \to \mathcal{O}^{\times}$, $\psi : G_{\mathfrak{n}'}^{+} \to \mathcal{O}^{\times}$. Recall

$$c(\mathfrak{m}, E_k(\psi, \chi)) = \sum_{\mathfrak{r}|\mathfrak{m}} \psi\left(\frac{\mathfrak{m}}{\mathfrak{r}}\right) \chi(\mathfrak{r}) N(\mathfrak{r})^{k-1}.$$

If $k > 1$,

$$c_\lambda(0, E_k(\psi, \chi)) = 2^{-n} \cdot \begin{cases} \psi^{-1}(\lambda) L(\psi^{-1}\chi, 1-k) & \text{if } \mathfrak{n} = 1 \\ 0 & \text{if } \mathfrak{n} \neq 1, \end{cases}$$

$$c_\lambda(0, E_1(\psi, \chi)) = 2^{-n} \cdot \begin{cases} \psi^{-1}(\lambda) L(\psi^{-1}\chi, 0) & \text{if } \mathfrak{n} = 1 \neq \mathfrak{n} \\ \chi^{-1}(\lambda) L(\psi\chi^{-1}, 0) & \text{if } \mathfrak{n}' = 1 \neq \mathfrak{n} \\ \psi^{-1}(\lambda) L(\chi\psi^{-1}, 0) \\ \quad + \chi^{-1}(\lambda) L(\psi\chi^{-1}, 0) & \text{if } \mathfrak{n} = \mathfrak{n}' = 1 \\ 0 & \text{if } \mathfrak{n} \neq 1 \neq \mathfrak{n}'. \end{cases}$$

Eisenstein series

Lemma

$$E_k(\psi,\chi) = E_k(\psi_{\mathfrak{p}},\chi) + \psi(\mathfrak{p})E_k(\psi,\chi)|_{\mathfrak{p}}.$$

Proof.

Assume $\mathfrak{p}|\mathfrak{m}$ (the other case is even easier):

$$c(\mathfrak{m}, E_k(\psi,\chi)) = \sum_{\mathfrak{r}|\mathfrak{m}} \psi(\mathfrak{r})\chi\left(\frac{\mathfrak{m}}{\mathfrak{r}}\right) N\left(\frac{\mathfrak{m}}{\mathfrak{r}}\right)^{k-1}$$

$$= \sum_{\substack{\mathfrak{r}|\mathfrak{m} \\ (\mathfrak{r},\mathfrak{p})=1}} \psi(\mathfrak{r})\chi\left(\frac{\mathfrak{m}}{\mathfrak{r}}\right) N\left(\frac{\mathfrak{m}}{\mathfrak{r}}\right)^{k-1} + \sum_{\mathfrak{r}|\frac{\mathfrak{m}}{\mathfrak{p}}} \psi(\mathfrak{r}\mathfrak{p})\chi\left(\frac{\mathfrak{m}}{\mathfrak{r}\mathfrak{p}}\right) N\left(\frac{\mathfrak{m}}{\mathfrak{r}\mathfrak{p}}\right)^{k-1}$$

$$= c(\mathfrak{m}, E_k(\psi_{\mathfrak{p}},\chi)) + \psi(\mathfrak{p})c(\mathfrak{m}, E_k(\psi,\chi)|_{\mathfrak{p}}).$$

Eisenstein series

Proof.

Assume $k > 1$, $\mathfrak{n} = 1$ (the case $\mathfrak{n} \neq 1$ is trivial):

$$c_\lambda(0, E_k(\psi, \chi)) = 2^{-n}\psi^{-1}(\lambda)L(\psi^{-1}\chi, 1-k);$$

$$c_\lambda(0, E_k(\psi_\mathfrak{p}, \chi)) = 0;$$

$$\begin{aligned}\psi(\mathfrak{p})c_\lambda(0, E_k(\psi, \chi)|_\mathfrak{p}) &= \psi(\mathfrak{p})c_{\lambda\mathfrak{p}}(0, E_k(\psi, \chi)) \\ &= \psi(\mathfrak{p})2^{-n}\psi^{-1}(\lambda\mathfrak{p})L(\psi^{-1}\chi, 1-k).\end{aligned}$$

When $k = 1$ there are some more cases. □

Eisenstein series

Lemma

$$E_k(\psi,\chi) = E_k(\psi,\chi_{\mathfrak{p}}) + \chi(\mathfrak{p})N(\mathfrak{p})^{k-1}E_k(\psi,\chi)|_{\mathfrak{p}}.$$

Proof.

Assume $\mathfrak{p}|\mathfrak{m}$ (the other case is even easier):

$$c(\mathfrak{m}, E_k(\psi,\chi)) = \sum_{\mathfrak{r}|\mathfrak{m}} \psi\left(\frac{\mathfrak{m}}{\mathfrak{r}}\right)\chi(\mathfrak{r})N(\mathfrak{r})^{k-1}$$

$$= \sum_{\substack{\mathfrak{r}|\mathfrak{m} \\ (\mathfrak{r},\mathfrak{p})=1}} \psi\left(\frac{\mathfrak{m}}{\mathfrak{r}}\right)\chi(\mathfrak{r})\,N(\mathfrak{r})^{k-1} + \sum_{\mathfrak{r}|\frac{\mathfrak{m}}{\mathfrak{p}}} \psi\left(\frac{\mathfrak{m}}{\mathfrak{r}\mathfrak{p}}\right)\chi(\mathfrak{r}\mathfrak{p})\,N(\mathfrak{r}\mathfrak{p})^{k-1}$$

$$= c(\mathfrak{m}, E_k(\psi,\chi_{\mathfrak{p}})) + \chi(\mathfrak{p})N(\mathfrak{p})^{k-1}c(\mathfrak{m}, E_k(\psi,\chi)|_{\mathfrak{p}}).$$

Eisenstein series

Proof.

Assume $k > 1$, $\mathfrak{n} = 1$ (the case $\mathfrak{n} \neq 1$ is trivial):

$$\begin{aligned} c_\lambda(0, E_k(\psi, \chi)) &= \frac{\psi^{-1}(\lambda)}{2^n} L(\psi^{-1}\chi, 1-k) \\ &= \frac{\psi^{-1}(\lambda)}{2^n} \sum_{\mathfrak{m}} \psi^{-1}\chi(\mathfrak{m}) N\mathfrak{m}^{k-1} \\ &= \frac{\psi^{-1}(\lambda)}{2^n} \sum_{(\mathfrak{m},\mathfrak{p})=1} \psi^{-1}\chi(\mathfrak{m}) N\mathfrak{m}^{k-1} \\ &\quad + \frac{\psi^{-1}(\lambda)}{2^n} \sum_{\mathfrak{m}} \psi^{-1}\chi(\mathfrak{p}\mathfrak{m}) N(\mathfrak{p}\mathfrak{m})^{k-1} \\ &= c_\lambda(0, E_k(\psi, \chi_\mathfrak{p})) + \chi(\mathfrak{p}) N(\mathfrak{p})^{k-1} c_\lambda(0, E_k(\psi, \chi)|_\mathfrak{p}). \end{aligned}$$

When $k = 1$ there are some more cases. □

Eisenstein series

Corollary

In particular we will use these formulas when $\chi = 1$:

$$E_k(\psi, 1) = E_k(\psi_{\mathfrak{p}}, 1) + \psi(\mathfrak{p}) E_k(\psi, 1)|_{\mathfrak{p}}$$

and

$$E_k(\psi, 1) = E_k(\psi, 1_{\mathfrak{p}}) + N(\mathfrak{p})^{k-1} E_k(\psi, 1)|_{\mathfrak{p}}.$$

Notation

Let

$$\psi : G_F \to \mathbb{C}^*$$

be totally odd of conductor $\mathfrak{c}_0$.
Let

$$T = \{\mathfrak{l}_1, \ldots, \mathfrak{l}_m\}$$

be a set of distinct primes, $\mathfrak{l}_i \nmid \mathfrak{c}_0$, let $\mathfrak{l} = \prod_{i=1}^m \mathfrak{l}_i$.
Let

$$\mathfrak{n} = \mathrm{cond}(H/F) \prod_{\mathfrak{q} \in T} \mathfrak{q}$$

and

$$\mathfrak{P} = p - \text{part of } \mathfrak{n}.$$

Let $G = \mathrm{Gal}(H/F) = G_p \times G'$, there G_p is a p-group and $\#G'$ is coprime to p. Let χ be a totally odd faithful character of G'.

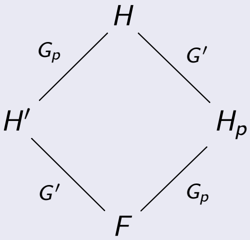
Lemma 8.13

Lemma 8.13

Let $\psi = \psi_p \chi$ be a character of R with $\mathrm{cond}(\psi) = \mathfrak{c}_0$. There exist distinct primes $\mathfrak{l}_i \nmid \mathfrak{c}_0 p$ and $\mathfrak{P}'$ divisible only by primes above p s.t.

$$\mathfrak{n} = \mathrm{cond}(H/F) \prod_{\mathfrak{q} \in T} \mathfrak{q} = \mathfrak{c}_0 \mathfrak{l}_1 \cdots \mathfrak{l}_d \mathfrak{P}'.$$

Proof.



To prove:

$\mathfrak{l} \nmid p$, $n \geq 2$, $\mathfrak{l}^n | \mathrm{cond}(H/F) \Rightarrow \mathfrak{l}^n | \mathrm{cond}(\psi)$.

Note: $\mathfrak{l}$ ramifies at most tamely in H_p/F, hence $\mathfrak{l}^n | \mathrm{cond}(H'/F)$.

Since χ is faithful on G', $\mathfrak{l}^n | \mathrm{cond}(\chi)$.

Hence $\mathfrak{l}^n | \mathrm{cond}(\psi)$.

□

Notation

It follows that $\mathfrak{c} := \mathfrak{c}_0\mathfrak{P}' = \mathrm{lcm}(\mathfrak{c}_0, \mathfrak{P})$. If $\mathfrak{l}' = \mathfrak{l}_1 \cdots \mathfrak{l}_d$, then $\mathfrak{n} = \mathfrak{c}\mathfrak{l}'$ is as in Werner's talk.
Recall:

Definition

$$W_k(\psi_{\mathfrak{P}}, 1) = \sum_{\mathfrak{m}'|\mathfrak{l}'} \mu(\mathfrak{m}')\psi(\mathfrak{m}')N\mathfrak{m}'^k E_k(\psi_{\mathfrak{P}}, 1)|_{\mathfrak{m}'} \in M_k(\mathfrak{n}, \psi).$$

We now interpolate this construction to a group ring family.

Proposition 8.14

Proposition 8.14

For all odd $k \geq 1$, let $W_k(\psi, 1) \in M_k(\mathfrak{n}, \mathrm{Frac}(R), \psi)$ be the unique form specializing to $W_k(\psi_{\mathfrak{P}}, 1)$ for all characters ψ of R. Then $c(\mathfrak{m}, W_k(\psi, 1)) \in R$ for all $0 \neq \mathfrak{m} \subseteq \mathcal{O}_F$.

Proof.

Idea: Define an explicit $W_k(\psi, 1) \in M_k(\mathfrak{n}, \mathrm{Frac}(R), \psi)$, which has $c(\mathfrak{m}, W_k(\psi, 1)) \in R$. Show that its specialization is $W_k(\psi_{\mathfrak{P}}, 1)$.
Notation: let $\mathfrak{l} = \prod_{\mathfrak{q}|\mathfrak{n}/\mathfrak{P}} \mathfrak{q}$, let $I_{\mathfrak{m}}$ be the product of the I_v for $v|\mathfrak{m}$.
Note: $G/I_{\mathfrak{m}}$ is the Galois group of the maximal subextension of H/F unramified at primes dividing $\mathfrak{m}$, hence a quotient of $G^+_{\mathfrak{n}/\mathfrak{m}}$.
Define: $\psi^{\mathfrak{m}} : G^+_{\mathfrak{n}/\mathfrak{m}} \to G/I_{\mathfrak{m}} \to \mathcal{O}[G/I_{\mathfrak{m}}]^*$.

Proposition 8.14

Proof.

Andreas: there is a $E_k(\psi^{\mathfrak{m}}, 1) \in \mathcal{M}_k(\mathfrak{n}/\mathfrak{m}, \psi, \mathcal{O}[G/I_{\mathfrak{m}}]^*)$ with specialization at a ψ (a character unramified at all primes dividing $\mathfrak{m}$) equal to $E_k(\psi_{\mathfrak{n}/\mathfrak{m}}, 1)$.
Next lift $x \in \mathcal{O}[G/I_{\mathfrak{m}}]$ to $\tilde{x} \in \mathcal{O}[G]$, let $NI_{\mathfrak{m}}$ act on it (so it does not depend on the lift) and project it to R.
The image of $E_k(\psi^{\mathfrak{m}}, 1)$ is $NI_{\mathfrak{m}} \cdot \tilde{E}_k(\psi^{\mathfrak{m}}, 1) \in \mathcal{M}_k(\mathfrak{n}/\mathfrak{m}, \mathrm{Frac}(R))$.
All of the non-constant q-expansion coefficients lie in R.
Combine them to

$$W_k(\psi) = \sum_{\mathfrak{m}|\mathfrak{l}} NI_{\mathfrak{m}} \cdot \tilde{E}_k(\psi^{\mathfrak{m}}, 1)|_{\mathfrak{m}} \psi^{\mathfrak{m}}(\mathfrak{m}) \frac{1}{\# I_{\mathfrak{m}}} \prod_{v|\mathfrak{m}} (1 - Nv^k).$$

All of the non-constant q-expansion coefficients lie in R.

Proposition 8.14

Proof.

It remains to show that it specializes to $W_k(\psi_{\mathfrak{P}}, 1)$.
Write $\mathfrak{n} = \mathfrak{c}_0 \mathfrak{l}' \mathfrak{P}'$ as in Lemma 8.13. Note that if ψ is nontrivial on $I_{\mathfrak{m}}$, then $\psi(NI_{\mathfrak{m}}) = 0$.

$$\psi(W_k(\psi,1)) = \psi\left(\sum_{\mathfrak{m}|\mathfrak{l}} NI_{\mathfrak{m}} \cdot \tilde{E}_k(\psi^{\mathfrak{m}},1)|_{\mathfrak{m}}\psi^{\mathfrak{m}}(\mathfrak{m})\frac{1}{\# I_{\mathfrak{m}}}\prod_{v|\mathfrak{m}}(1-Nv^k)\right)$$

$$= \sum_{\mathfrak{m}|\mathfrak{l}'} \psi(\mathfrak{m}) E_k(\psi_{\frac{\mathfrak{l}'}{\mathfrak{m}}\mathfrak{P}},1)|_{\mathfrak{m}} \sum_{\mathfrak{m}'|\mathfrak{m}} \mu(\mathfrak{m}') N\mathfrak{m}'^k$$

$$= \sum_{\mathfrak{m}'|\mathfrak{l}'} \mu(\mathfrak{m}') N\mathfrak{m}'^k \psi(\mathfrak{m}') \left(\sum_{\mathfrak{m}\left|\frac{\mathfrak{l}'}{\mathfrak{m}'}\right.} \psi(\mathfrak{m}) E_k(\psi_{\frac{\mathfrak{l}'}{\mathfrak{m}'\mathfrak{m}}\mathfrak{P}},1)|_{\mathfrak{m}}\right)|_{\mathfrak{m}'}.$$

Proposition 8.14

Proof.

It remains to prove that $\sum_{\mathfrak{m}|\frac{\mathfrak{l}'}{\mathfrak{m}'}} \psi(\mathfrak{m})E_k(\psi_{\frac{\mathfrak{l}'}{\mathfrak{m}'\mathfrak{m}}\mathfrak{P}}, 1)|_{\mathfrak{m}} = E_k(\psi_{\mathfrak{P}}, 1)$.

Induction on the number of prime factors of $\mathfrak{l}'/\mathfrak{m}' = \mathfrak{l}_1 \cdots \mathfrak{l}_r$. The case $r = 1$ is clear.

$$\sum_{\mathfrak{m}|\mathfrak{l}_1\cdots\mathfrak{l}_r} \psi(\mathfrak{m})E_k(\psi_{\frac{\mathfrak{l}_1\cdots\mathfrak{l}_r}{\mathfrak{m}}\mathfrak{P}}, 1)|_{\mathfrak{m}}$$

$$= \sum_{\mathfrak{m}|\mathfrak{l}_1\cdots\mathfrak{l}_{r-1}} \psi(\mathfrak{m})(E_k(\psi_{\frac{\mathfrak{l}_1\cdots\mathfrak{l}_r}{\mathfrak{m}}\mathfrak{P}}, 1)|_{\mathfrak{m}} + \psi(\mathfrak{l}_r)E_k(\psi_{\frac{\mathfrak{l}_1\cdots\mathfrak{l}_{r-1}}{\mathfrak{m}}\mathfrak{P}}, 1)|_{\mathfrak{m}\mathfrak{l}_r})$$

$$= \sum_{\mathfrak{m}|\mathfrak{l}_1\cdots\mathfrak{l}_{r-1}} \psi(\mathfrak{m})E_k(\psi_{\frac{\mathfrak{l}_1\cdots\mathfrak{l}_{r-1}}{\mathfrak{m}}\mathfrak{P}}, 1)|_{\mathfrak{m}} = E_k(\psi_{\mathfrak{P}}, 1),$$

where the last equality holds by the induction hypothesis. □

J. Silliman, Group ring valued Hilbert modular forms, Thm. 8.2

Theorem 8.15 (Silliman)

Let $m' \geq \#G_p$, k big enough and odd. Let $f_\psi \in M_k(\mathfrak{n}, E, \psi)$, whose normalized constant terms at representatives for each cusp $A \in C_\infty(\mathfrak{P}, \mathfrak{n})$ are divisible by $p^{m'}$.
Then $\exists h(\psi) \in M_k(\mathfrak{n}, \psi, R)$ such that

$$\tilde{f}_\psi = f_\psi - (p^{m'}/\#G_p)h(\psi)$$

has constant term 0 at all cusps $A \in C_\infty(\mathfrak{P}, \mathfrak{n})$.
If $\mathfrak{P} = 1$, so $C_\infty(\mathfrak{P}, \mathfrak{n}) = \mathrm{cusps}(\mathfrak{n})$, then $\tilde{f}_\psi$ is cuspidal.
If $\mathfrak{P} \neq 1$, then $e_{\mathfrak{P}}^{\mathrm{ord}}(\tilde{f}_\psi)$ is cuspidal.

Lemma 8.16

Lemma 8.16

Assume no prime above p ramifies in H/F (case 1), N large enough, $k \equiv 1 \pmod{(p-1)p^N}$ positive. Let

$$x = \frac{\Theta_{S_\infty}(1-k)}{\Theta_{S_\infty}(0)} \in \mathrm{Frac}(R).$$

Then $x \in R$, it is a non-zerodivisor and

$$x \equiv \prod_{\mathfrak{p}|p} (1 - \chi(\mathfrak{p})^{-1}) \pmod{\mathfrak{m}_R}.$$

Proof.

The specializations to ψ of numerator and denominator are $L(\psi^{-1}, 1-k)$ and $L(\psi^{-1}, 0)$ respectively, which are nonzero. So x is well-defined and if it is in R it is a non-zerodivisor.

Lemma 8.16

Proof.

Let $S_p = S_\infty \cup \{\text{primes above } p\}$, $\sigma_\mathfrak{p}$ the Frobenius (case 1!).

$$\Theta_{S_p}(1-k) = \prod_{\mathfrak{p}|p}(1-\sigma_\mathfrak{p}^{-1}N\mathfrak{p}^{k-1})\Theta_{S_\infty}(1-k).$$

Let m' be such that $p^{m'-1}/\Theta_{S_\infty}(0) \in R$. By continuity of the L-functions, for $k \equiv 1 \pmod{(p-1)p^N}$ and N large enough, $y(k) = \Theta_{S_p}(1-k) - \Theta_{S_p}(0)$ is divisible by $p^{m'}$ in R. Then

$$\frac{y(k)}{\Theta_{S_\infty}(0)} = x\prod_{\mathfrak{p}|p}(1-\sigma_\mathfrak{p}^{-1}N\mathfrak{p}^{k-1}) - \prod_{\mathfrak{p}|p}(1-\sigma_\mathfrak{p}^{-1}) \in \mathfrak{m}_R.$$

For $k > 1$, $\prod_{\mathfrak{p}|p}(1-\sigma_\mathfrak{p}^{-1}N\mathfrak{p}^{k-1})$ is a unit and its specializations at ψ are congruent to 1 modulo $\mathfrak{p}$. □

Theorem 8.17

Theorem 8.17

Assume no prime above p ramifies in H/F (case 1), N large enough, $k \equiv 1 \pmod{(p-1)p^N}$ positive. Then there exists $H_k(\psi) \in M_k(\mathfrak{n}, R, \psi)$ such that

$$\tilde{F}_k(\psi) = xW_1(\psi, 1)V_{k-1} - W_k(\psi, 1) - x\Theta^{\#}(0)H_k(\psi)$$

lies in $S_k(\mathfrak{n}, R, \psi)$, where x is as in Lemma 8.16.

Proof.

Define $f_k(\psi) \in M_k(\mathfrak{n}, \mathrm{Frac}(R), \psi)$

$$f_k(\psi) = W_1(\psi, 1)V_{k-1} - \frac{1}{x}W_k(\psi, 1) - \frac{\Theta^{\#}(0)}{2^n}G_k(\psi).$$

By Prop. 8.11 (Werner) the constant terms are divisible by $p^{m'}$.

Theorem 8.17

Proof.

By Theorem 8.15 ([Silliman]) $\exists h_k(\psi) \in M_k(\mathfrak{n}, R, \psi)$ such that $\tilde{f}_k(\psi) = f_k(\psi) - \frac{p^{m'}}{\#G_p} h_k(\psi)$ is a cusp form.

Choose m' large enough that $\#G_p \cdot \Theta^{\#}(0)$ divides $p^{m'}$ in R and define

$$H_k(\psi) = \frac{G_k(\psi)}{2^n} + \frac{p^{m'}}{\#G_p \cdot \Theta^{\#}(0)} h_k(\psi) \in M_k(\mathfrak{n}, R, \psi).$$

Define the cusp form $\tilde{F}_k(\psi) = x \cdot \tilde{f}_k(\psi)$,

$$\tilde{F}_k(\psi) = xW_1(\psi, 1)V_{k-1} - W_k(\psi, 1) - x\Theta^{\#}(0)H_k(\psi).$$

Actually the constant terms of $W_1(\psi, 1)$ are not in R, but nonconstant terms in V_{k-1} are highly divisible by p. □

Theorem 8.18

Theorem 8.18

Assume at least one prime above p ramifies in H/F (case 2), N large enough, $k \equiv 1 \pmod{(p-1)p^N}$ positive. Then there exists $H_k(\psi) \in M_k(\mathfrak{n}, R, \psi)$ such that

$$\tilde{F}_k(\psi) = e_{\mathfrak{P}}^{\mathrm{ord}}(W_1(\psi, 1)V_{k-1} - \Theta^{\#}(0)H_k(\psi))$$

lies in $S_k(\mathfrak{n}, R, \psi)$.

Proof.

As for the previous Theorem, but using Proposition 8.12 in place of Proposition 8.11. □

Applying the ordinary operator

Recall from Andreas:

$$T_{\mathfrak{l}} E_k(\psi_{\mathfrak{p}}, 1) = (\psi_{\mathfrak{p}}(\mathfrak{l}) + N(\mathfrak{l})^{k-1}) \cdot E_k(\psi_{\mathfrak{p}}, 1),$$

$$U_{\mathfrak{p}} E_k(\psi_{\mathfrak{p}}, 1) = N(\mathfrak{p})^{k-1} \cdot E_k(\psi_{\mathfrak{p}}, 1),$$

$$U_{\mathfrak{p}} E_k(\psi, 1)|_{\mathfrak{p}} = E_k(\psi, 1),$$

$$U_{\mathfrak{p}} E_k(\psi, 1_{\mathfrak{p}}) = \psi(\mathfrak{p}) E_k(\psi, 1_{\mathfrak{p}}) \Rightarrow e_{\mathfrak{p}}^{\mathrm{ord}} E_k(\psi, 1_{\mathfrak{p}}) = E_k(\psi, 1_{\mathfrak{p}}).$$

In particular:

$$T_{\mathfrak{l}} E_1(\psi_{\mathfrak{p}}, 1) = (\psi_{\mathfrak{p}}(\mathfrak{l}) + 1) \cdot E_1(\psi_{\mathfrak{p}}, 1),$$

$$U_{\mathfrak{p}} E_1(\psi_{\mathfrak{p}}, 1) = E_1(\psi_{\mathfrak{p}}, 1) \Rightarrow e_{\mathfrak{p}}^{\mathrm{ord}} E_1(\psi_{\mathfrak{p}}, 1) = E_1(\psi_{\mathfrak{p}}, 1).$$

Applying the ordinary operator

Applying $e_{\mathfrak{p}}^{\mathrm{ord}} U_{\mathfrak{p}}$ to

$$E_k(\psi, 1) = E_k(\psi, 1_{\mathfrak{p}}) + N(\mathfrak{p})^{k-1} E_k(\psi, 1)|_{\mathfrak{p}}$$

we obtain

$$e_{\mathfrak{p}}^{\mathrm{ord}}(U_{\mathfrak{p}} - N(\mathfrak{p})^{k-1}) E_k(\psi, 1) = \psi(\mathfrak{p}) E_k(\psi, 1_{\mathfrak{p}}).$$

If $k > 1$, then $U_{\mathfrak{p}} - N(\mathfrak{p})^{k-1}$ is invertible on the ordinary part and

$$e_{\mathfrak{p}}^{\mathrm{ord}} E_k(\psi, 1) = \frac{\psi(\mathfrak{p})}{\psi(\mathfrak{p}) - N(\mathfrak{p})^{k-1}} E_k(\psi, 1_{\mathfrak{p}}).$$

So in particular for $k > 1$

$$e_{\mathfrak{p}}^{\mathrm{ord}} E_k(\psi_{\mathfrak{p}}, 1) = 0.$$

Applying the ordinary operator

Applying $e_{\mathfrak{p}}^{\mathrm{ord}} U_{\mathfrak{p}}$ to

$$E_k(\psi, 1) = E_k(\psi_{\mathfrak{p}}, 1) + \psi(\mathfrak{p}) E_k(\psi, 1)|_{\mathfrak{p}}$$

we obtain

$$e_{\mathfrak{p}}^{\mathrm{ord}}(U_{\mathfrak{p}} - \psi(\mathfrak{p})) E_k(\psi, 1) = e_{\mathfrak{p}}^{\mathrm{ord}} U_{\mathfrak{p}} E_k(\psi_{\mathfrak{p}}, 1),$$

which is 0 for $k > 1$.

Corollary 8.19

Corollary 8.19

Assume no prime above p ramifies in H/F (case 1), N large enough, $k \equiv 1 \pmod{(p-1)p^N}$ positive; let $\mathfrak{P}'$ be the product of the primes above p. Then $\exists F_k(\psi) \in S_k(\mathfrak{n}\mathfrak{P}', \psi, R)^{p-\mathrm{ord}}$ such that

$$F_k(\psi) \equiv \begin{cases} xW_1(\psi, 1) - W_k(\psi, 1_p) \pmod{x\Theta^\#} & \exists \mathfrak{p}|p \ \chi(\mathfrak{p}) = 1 \\ W_1(\psi_p, 1) \pmod{\Theta^\#} & \forall \mathfrak{p}|p \ \chi(\mathfrak{p}) \neq 1. \end{cases}$$

Corollary 8.19

Proof.

Case 1a: $\exists \mathfrak{p} | p \ \chi(\mathfrak{p}) = 1$. From

$$e_{\mathfrak{p}}^{\mathrm{ord}} E_k(\psi, 1) = \frac{\psi(\mathfrak{p})}{(\psi(\mathfrak{p}) - N(\mathfrak{p})^{k-1})} E_k(\psi, 1_{\mathfrak{p}}).$$

we get

$$e_{\mathfrak{P}'}^{\mathrm{ord}} W_k(\psi, 1) = z \cdot W_k(\psi, 1_{\mathfrak{p}}),$$

where

$$z = \prod_{\mathfrak{p}|p} \frac{\psi(\mathfrak{p})}{(\psi(\mathfrak{p}) - N(\mathfrak{p})^{k-1})} \equiv 1 \pmod{p^m}.$$

Corollary 8.19

Proof.

From Theorem 8.17

$$\tilde{F}_k(\psi) = xW_1(\psi,1)V_{k-1} - W_k(\psi,1) - x\Theta^{\#}(0)H_k(\psi).$$

Then

$$\begin{aligned} F_k(\psi) &= z^{-1} e_{\mathfrak{P}'}^{\mathrm{ord}} \tilde{F}_k(\psi) \\ &\equiv z^{-1} e_{\mathfrak{P}'}^{\mathrm{ord}} xW_1(\psi_{\mathfrak{p}},1)V_{k-1} - W_k(\psi,1_p) \pmod{x\Theta^{\#}}. \end{aligned}$$

Actually we would like

$$\equiv xW_1(\psi,1) - W_k(\psi,1_p) \pmod{x\Theta^{\#}}.$$

Corollary 8.19

Proof.

Case 1b: $\forall \mathfrak{p} | p \ \chi(\mathfrak{p}) \neq 1$.
Applying $U_\mathfrak{p}$ to $E_1(\psi, 1) = E_1(\psi_\mathfrak{p}, 1) + \psi(\mathfrak{p}) E_1(\psi, 1)|_\mathfrak{p}$ we obtain

$$U_\mathfrak{p} E_1(\psi, 1) = E_1(\psi_\mathfrak{p}, 1) + \psi(\mathfrak{p}) E_1(\psi, 1),$$

hence

$$(U_\mathfrak{p} - \psi(\mathfrak{p})) E_1(\psi, 1) = E_1(\psi_\mathfrak{p}, 1)$$

and

$$e_\mathfrak{p}^{\mathrm{ord}} (U_\mathfrak{p} - \psi(\mathfrak{p})) E_1(\psi, 1) = E_1(\psi_\mathfrak{p}, 1).$$

Corollary 8.19

Proof.

Therefore

$$e_{\mathfrak{P}'}^{\mathrm{ord}} \prod_{\mathfrak{p}|p} (U_{\mathfrak{p}} - \psi(\mathfrak{p}))(W_1(\psi, 1)) = W_1(\psi_p, 1).$$

Dasgupta and Kakde say there is a factor $z = \frac{1}{1-\psi(\mathfrak{p})} \in R^*$.
For $k > 0$,

$$e_{\mathfrak{P}'}^{\mathrm{ord}} \prod_{\mathfrak{p}|p} (U_{\mathfrak{p}} - \psi(\mathfrak{p}))(W_k(\psi, 1)) = 0.$$

Corollary 8.19

Proof.

To conclude, set

$$F_k(\psi) = (xz)^{-1} e^{\mathrm{ord}}_{\mathfrak{P}'} \prod_{\mathfrak{p}|p} (U_{\mathfrak{p}} - \psi(\mathfrak{p}))(\tilde{F}_k(\psi)).$$

□

Remark 8.20

The congruences Corollary 8.19 should be seen as congruences of Fourier coefficients. For all ideals $\mathfrak{m}$ case 1a and 1b become:

$$c(\mathfrak{m}, F_k(\psi)) \equiv x \cdot c(\mathfrak{m}, W_1(\psi, 1)) - c(\mathfrak{m}, W_k(\psi, 1_p)) \pmod{x\Theta^{\#}},$$

$$c(\mathfrak{m}, F_k(\psi)) \equiv c(\mathfrak{m}, W_1(\psi_p, 1)) \pmod{\Theta^{\#}}.$$

Corollary 8.21

Corollary 8.21

Assume at least one prime above p ramifies in H/F (case 2), N large enough, $k \equiv 1 \pmod{(p-1)p^N}$ positive. Let $\mathfrak{P}'$ denote the product of the primes above p that do not divide $\mathfrak{n}$, $\mathfrak{P}''$ the product of the primes $\mathfrak{p}$ dividing $\mathfrak{P}'$ such that $\chi(\mathfrak{p}) \neq 1$. Then $\exists F_k(\psi) \in S_k(\mathfrak{n}\mathfrak{P}', \psi, R)^{p-\mathrm{ord}}$ such that

$$F_k(\psi) \equiv W_1(\psi_{\mathfrak{P}\mathfrak{P}''}, 1) \pmod{\Theta^\#}.$$

Notation

$$\mathfrak{n} = \mathrm{cond}(H/F) \prod_{\mathfrak{q} \in T} \mathfrak{q}$$

$$\mathfrak{P} = \mathrm{gcd}(p^\infty, \mathfrak{n})$$

$$\mathfrak{P}' = \prod_{\mathfrak{p} | p, \ \mathfrak{p} \nmid \mathfrak{P}} \mathfrak{p}$$

$$\mathfrak{P}'' = \prod_{\mathfrak{p} | \mathfrak{P}', \ \chi(\mathfrak{p}) \neq 1} \mathfrak{p}.$$

Case 1a: $\mathfrak{P} = 1, \ \mathfrak{P}' \neq \mathfrak{P}''$,

Case 1b: $\mathfrak{P} = 1, \ \mathfrak{P}' = \mathfrak{P}''$,

Case 2: $\mathfrak{P} \neq 1$.

Hecke algebra

Definition

Let

$$\tilde{\boldsymbol{T}} \subset \mathrm{End}_R(S_k(\mathfrak{n}\mathfrak{P}', R, \psi)^{p-\mathrm{ord}})$$

denote the Hecke algebra of $S_k(\mathfrak{n}\mathfrak{P}', R, \psi)^{p-\mathrm{ord}}$ generated over R by the operators $T_{\mathfrak{l}}$ for $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{P}'$, $U_{\mathfrak{p}}$ for $\mathfrak{p}|p$ and the diamond operators $S(\mathfrak{m})$. Note that $S(\mathfrak{m})$ simply act by $\psi(\mathfrak{m}) \in R^*$.

Definition

Let $\boldsymbol{T} \subset \tilde{\boldsymbol{T}}$ denote the sub-R-algebra generated by $T_{\mathfrak{l}}$ for $\mathfrak{l} \nmid \mathfrak{n}\mathfrak{P}'$, $U_{\mathfrak{p}}$ for $\mathfrak{p}|\mathfrak{P}$ and $S(\mathfrak{m})$. In other words, the operators $U_{\mathfrak{p}}$ for $\mathfrak{p}|\mathfrak{P}'$ are excluded.

Hecke algebra

Definition

Let M denote the set of p-ordinary cuspidal newforms of weight k, level dividing $\mathfrak{m}\mathfrak{P}'$ and nebentypus ψ for all $\psi \in \Psi$, where $R = R_\Psi$. For each $f \in M$ denote by f_p the ordinary stabilization of f with respect to all primes $\mathfrak{p}|p$.

Injections

There are injections with finite cokernels

$$\boldsymbol{T} \hookrightarrow \tilde{\boldsymbol{T}} \hookrightarrow \prod_M \mathcal{O}$$

that send $T_\mathfrak{l} \mapsto (c(\mathfrak{l}, f_p))_{f\in M}$ and $U_\mathfrak{p} \mapsto (c(\mathfrak{p}, f_p))_{f\in M}$.

Eigenvectors

Eigenvectors

$$F_k(\psi)|_{T_\mathfrak{l}} \equiv \begin{cases} (\psi(\mathfrak{l}) + \varepsilon_{\mathrm{cyc}}^{k-1}(\mathfrak{l}))F_k(\psi) \pmod{x\Theta^{\#}} & \text{in case 1} \\ (\psi(\mathfrak{l}) + 1)F_k(\psi) \pmod{\Theta^{\#}} & \text{in case 2,} \end{cases}$$

with $\varepsilon_{\mathrm{cyc}}(\mathfrak{l}) = \langle N\mathfrak{l} \rangle \in \boldsymbol{Z}_p^*$ for $\mathfrak{l} \nmid p$. For $\mathfrak{p}|\mathfrak{P}\mathfrak{P}''$:

$$F_k(\psi)|_{U_\mathfrak{l}} \equiv F_k(\psi) \begin{cases} \pmod{x\Theta^{\#}} & \text{in case 1} \\ \pmod{\Theta^{\#}} & \text{in case 2.} \end{cases}$$

Lemma 8.22

Lemma 8.22

Let k_E be the residue field of $\mathcal{O} = \mathcal{O}_E$. There is an $\mathcal{O}$-algebra homomorphism $\bar{\varphi} : \boldsymbol{T} \to k_E$ given by

- $\bar{\varphi}(T_{\mathfrak{l}}) = 1 + \chi(\mathfrak{l})$ for $\mathfrak{l} \nmid \mathfrak{n}p$,
- $\bar{\varphi}(U_{\mathfrak{p}}) = 1$ for $\mathfrak{p}|\mathfrak{P}$,
- $\bar{\varphi}(S(\mathfrak{m})) = \chi(\mathfrak{m})$.

Proof.

$\bar{\varphi}$ sends each operator to the corresponding eigenvalue of $F_k(\psi)$ modulo the maximal ideal $\mathfrak{m}_R$ of R; note that $R/\mathfrak{m}_R \cong k_E$. □

Maximal Eisenstein ideal

Maximal Eisenstein ideal

Let $\mathfrak{m} \subset \boldsymbol{T}$ be the kernel of $\bar{\varphi}$. Let $\boldsymbol{T}_{\mathfrak{m}}$ and $\tilde{\boldsymbol{T}}_{\mathfrak{m}} = \tilde{\boldsymbol{T}} \otimes_{\boldsymbol{T}} \boldsymbol{T}_{\mathfrak{m}}$ the $\mathfrak{m}$-adic completions of $\boldsymbol{T}$ and $\tilde{\boldsymbol{T}}$. Let $\bar{M} \subset M$ denote the set of $f \in M$ such that $c(1, (f_p)|_t) \equiv \bar{\varphi}(t) \pmod{\pi_E}$ for all $t \in \boldsymbol{T}$. Then

$$\left(\prod_{f \in M} \mathcal{O}\right)_{\mathfrak{m}} = \prod_{f \in \bar{M}} \mathcal{O}.$$

The Artin-Rees Lemma yields injections with finite cokernel

$$\boldsymbol{T}_{\mathfrak{m}} \hookrightarrow \tilde{\boldsymbol{T}}_{\mathfrak{m}} \hookrightarrow \prod_{f \in \bar{M}} \mathcal{O}.$$

Theorem 8.23

Theorem 8.23

There is $x \in R$, an $R/x\Theta^{\#}$-algebra W and a surjective R-algebra homomorphism $\varphi : \tilde{\mathbf{T}}_{\mathfrak{m}} \to W$ such that

- The structure map $R/x\Theta^{\#} \to W$ is injective.
- $\varphi(\mathbf{T}_{\mathfrak{m}}) \subset R/x\Theta^{\#} \subset W$. More precisely

$$\varphi(S(\mathfrak{m})) = \psi(\mathfrak{m}) \text{ for } \mathfrak{m} \in G_{\mathfrak{n}}^{+},$$

$$\varphi(U_{\mathfrak{p}}) = 1 \text{ for } \mathfrak{p}|\mathfrak{P}, \;\; \varphi(T_{\mathfrak{l}}) = \varepsilon_{\mathrm{cyc}}^{k-1}(\mathfrak{l}) + \psi(\mathfrak{l}) \text{ for } \mathfrak{l} \nmid \mathfrak{n}p.$$

- Let

$$\tilde{U} = \prod_{\mathfrak{p}|\mathfrak{P}'} (U_{\mathfrak{p}} - \psi(\mathfrak{p})) \in \tilde{\mathbf{T}}_{\mathfrak{m}},$$

  and $U = \varphi(\tilde{U})$. If $y \in R$ and $Uy = 0$ in W, then $y \in (\Theta^{\#})$.

Theorem 8.23

Proof

We only consider the case 1a with x as in Theorem 8.17. Let $\mathcal{C} = \prod_{\mathfrak{a}\subset\mathcal{O}_F} R/x\Theta^{\#}$. There is an R-module homomorphism $c : S_k(\mathfrak{n}, R, \psi) \to \mathcal{C}$, $f \mapsto (c(\mathfrak{a}, f))_{\mathfrak{a}}$. Define an action of the Hecke operators on $\mathcal{C}$, so that c is Hecke equivariant.
Let $\mathcal{F}$ denote the image of the $\tilde{\boldsymbol{T}}$-span of $F_k(\psi)$ under c.
Let W be the image of $\tilde{\boldsymbol{T}} \to \mathrm{End}_{R/c\Theta^{\#}}(\mathcal{F})$.
We get a surjective $\varphi_{\mathfrak{m}} : \tilde{\boldsymbol{T}} \to W$.
Recall that for $\mathfrak{l} \nmid \mathfrak{n}p$,

$$F_k(\psi)|_{T_{\mathfrak{l}}} \equiv (\varepsilon_{\mathrm{cyc}}^{k-1}(\mathfrak{l}) + \psi(\mathfrak{l}))F_k(\psi) \pmod{x\Theta^{\#}}.$$

Hence

$$\varphi(T_{\mathfrak{l}}) = \varepsilon_{\mathrm{cyc}}^{k-1}(\mathfrak{l}) + \psi(\mathfrak{l}).$$

Similarly for the other operators.

Theorem 8.23

Proof

Assume $\alpha \in R$ has vanishing image in W, i.e. $\alpha F_k(\psi) \equiv 0 \pmod{x\Theta^{\#}}$. Recall

$$F_k(\psi) \equiv xW_1(\psi, 1) - W_k(\psi, 1_p) \pmod{x\Theta^{\#}}.$$

Looking at the Fourier coefficients $c(\mathfrak{m}, \cdot)$ for $\mathfrak{m} = 1$ and $\mathfrak{m} = \mathfrak{p}$ for any $\mathfrak{p}|p$ we get

$$\begin{aligned} (x-1)\alpha &\equiv 0 \pmod{x\Theta^{\#}} \\ ((1+\psi(\mathfrak{p}))x - \psi(\mathfrak{p}))\alpha &\equiv 0 \pmod{x\Theta^{\#}}. \end{aligned}$$

Multiplying the first by $1 + \psi(\mathfrak{p})$ and subtracting it from the second yields $\alpha \equiv 0 \pmod{x\Theta^{\#}}$. This establishes the injectivity of $R/x\Theta^{\#} \to W$.

Theorem 8.23

Proof.

Dasgupta and Kakde say that

$$F_k(\psi)|_{\tilde{U}} \equiv xW_1(1, \psi_p) \pmod{x\Theta^\#}.$$

Therefore if $yF_k(\psi)|_{\tilde{U}} \equiv 0 \pmod{x\Theta^\#}$ for $y \in R$, considering the Fourier coefficient at $\mathfrak{m} = 1$, we see that $xy \in (x\Theta^\#)$ and hence $y \in (\Theta^\#)$ since x is a non-zerodivisor in R.
Cases 1b and 2 are similar. □

Thank you for your attention!