INVERSE PROBLEMS FOR NON-SELFADJOINT STURM-LIOUVILLE OPERATORS WITH DIRICHLET BOUNDARY CONDITIONS

S.A. Buterin

Abstract. An inverse spectral problem is studied for a non-selfadjoint Sturm-Liouville operator on a finite interval with an arbitrary behavior of the spectrum. The spectral data introduced generalize the classical discrete spectral data corresponding to the specification of the spectral function in the selfadjoint case. The connection with other types of spectral characteristics is investigated and by the method of spectral mappings a uniqueness theorem is proved. A constructive procedure for solving the inverse problem is obtained along with necessary and sufficient conditions of its solvability. Special attention is paid to the selfadjoint case and to the cases of both finite-dimensional and small perturbations of the spectral data. The stability of the inverse problem is proved.

2000 Mathematics Subject Classification: 34A55 34B24 47E05

Key words: non-selfadjoint Sturm-Liouville operators; Dirichlet boundary conditions; inverse spectral problems; method of spectral mappings; generalized weight numbers

1. INTRODUCTION

Consider the following boundary value problem \( L = L(q(x), T) : \)
\[
\ell y := -y'' + q(x)y = \lambda y, \quad 0 < x < T, \tag{1}
\]
\[
y(0) = y(T) = 0, \tag{2}
\]
where \( q(x) \in L_1(0, T) \) is a complex-valued function. Let \( \{\lambda_n\}_{n \geq 1} \) be the spectrum of \( L \). Denote by \( S(x, \lambda) \) a solution of equation (1) satisfying the initial conditions
\[
S(0, \lambda) = 0, \quad S'(0, \lambda) = 1. \tag{3}
\]

For the selfadjoint case, i.e. when \( q(x) \) is real, the inverse problem of recovering the Sturm-Liouville operator from its spectral characteristics was investigated fairly completely (see [1]–[6] and the references therein). In particular, applying the transformation operator Marchenko [2] proved that a selfadjoint Sturm-Liouville operator on the half-line or a finite interval is determined uniquely by specifying its spectral function. For a finite interval this corresponds to the specification of the classical discrete spectral data \( \{\lambda_n, \alpha_n\}_{n \geq 1} \), where \( \alpha_n \) are weight numbers defined by the formula
\[
\alpha_n = \int_0^T S^2(x, \lambda_n) \, dx. \tag{4}
\]
In the non-selfadjoint case the specification of \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) also determines the operator uniquely if the spectrum is simple. For the Robin boundary conditions the corresponding fact was shown in [6]. For the case of the multiple spectrum in [7], [8] generalized weight numbers were introduced, which are connected in a natural way with the coefficients of the main parts of the Weyl function in neighborhoods of the eigenvalues.

\[1\] This research was supported in part by Russian Foundation for Basic Research and Taiwan National Science Council (projects 10-01-00099 and 10-01-92001-NSC-a) and by Mikhail Lomonosov Program of the Ministry of Education and Science of Russia and DAAD (project 5007).
In the present paper generalized weight numbers are constructed for the case of the Dirichlet boundary conditions. The connection with other types of spectral characteristics, namely with the Weyl function and with the two spectra, is investigated. Developing the ideas of the method of spectral mappings \[6\], \[9\], \[10\] we prove that the specification of the generalized spectral data uniquely determines the potential \( q(x) \) and provide constructive procedures for solving the inverse problem together with necessary and sufficient conditions of its solvability. In the general non-selfadjoint case these conditions include the requirement of the solvability of the main equation of the inverse problem. In the last section the main cases are studied when the solvability of the main equation can be proved by sufficiency: selfadjoint case, the cases of finite-dimensional and small perturbations of the spectral data. The stability of the inverse problem is proved.

2. GENERALIZED SPECTRAL DATA. INVERSE PROBLEM

Let the function \( \psi(x, \lambda) \) be a solution of equation (1) under the conditions

\[
\psi(T, \lambda) = 0, \quad \psi'(T, \lambda) = -1.
\]

For every fixed \( x \in [0, T] \) the functions \( S(x, \lambda), \psi(x, \lambda) \) and their derivatives with respect to \( x \) are entire in \( \lambda \). The eigenvalues \( \lambda_n, \ n \geq 1, \) of the problem \( L \) coincide with the zeros of its characteristic function

\[
\Delta(\lambda) := \langle \psi(x, \lambda), S(x, \lambda) \rangle = \psi(0, \lambda) = S(T, \lambda),
\]

where \( \langle y, z \rangle := y'z - yz' \). It is known (see, e.g., \[6\]) that the spectrum \( \{\lambda_n\}_{n \geq 1} \) has the asymptotics

\[
\rho_n := \sqrt{\alpha_n} = \frac{\pi n}{T} + \frac{\omega}{\pi n} + \frac{\kappa_n}{n}, \quad \kappa_n = o(1),
\]

where

\[
\omega = Q(T), \quad Q(x) = \frac{1}{2} \int_0^x q(t) \, dt.
\]

Denote by \( m_n \) the multiplicity of the eigenvalue \( \lambda_n \) (\( \lambda_n = \lambda_{n+1} = \ldots = \lambda_{n+m_n-1} \)) and put \( S = \{n : n - 1 \in \mathbb{N}, \lambda_{n-1} \neq \lambda_n\} \cup \{1\} \). Note that by virtue of (7) for sufficiently large \( n \) we have \( m_n = 1 \). Denote

\[
S_\nu(x, \lambda) = \frac{1}{\nu!} \frac{d^\nu}{d\lambda^\nu} S(x, \lambda), \quad \psi_\nu(x, \lambda) = \frac{1}{\nu!} \frac{d^\nu}{d\lambda^\nu} \psi(x, \lambda).
\]

Hence, for \( \nu \geq 1, \ n \in S \) we have

\[
\begin{align*}
\ell S_\nu(x, \lambda) &= \lambda S_\nu(x, \lambda) + S_{\nu-1}(x, \lambda), \quad S_\nu(0, \lambda) = S'_\nu(0, \lambda) = 0, \\
\ell \psi_\nu(x, \lambda) &= \lambda \psi_\nu(x, \lambda) + \psi_{\nu-1}(x, \lambda), \quad \psi_\nu(T, \lambda) = \psi'_\nu(T, \lambda) = 0.
\end{align*}
\]

Moreover, (6) yields

\[
S_\nu(T, \lambda_n) = \psi_\nu(0, \lambda_n) = \frac{1}{\nu!} \Delta^{(\nu)}(\lambda_n) = 0, \quad \nu = 0, m_n - 1.
\]

Put

\[
S_{n+\nu}(x) = S_\nu(x, \lambda_n), \quad \psi_{n+\nu}(x) = \psi_\nu(x, \lambda_n), \quad n \in S, \nu = 0, m_n - 1.
\]
Thus, \( \{S_n(x)\}_{n \geq 1}, \{\psi_n(x)\}_{n \geq 1} \) are complete systems of eigen- and associated functions of the boundary value problem \( L \). Together with the eigenvalues \( \lambda_n \) we consider generalized weight numbers \( \alpha_n, \; n \geq 1 \), determined in the following way:

\[
\alpha_{k+\nu} = \int_0^T S_{k+\nu}(x)S_{k+m_k-1}(x) \, dx, \quad k \in \mathbb{S}, \; \nu = 0, m_k - 1.
\]  \( \text{(11)} \)

We note that the numbers \( \alpha_n \) for sufficiently large \( n \) coincide with the classical weight numbers \( (4) \) for the selfadjoint Sturm-Liouville operator.

**Definition 1.** The numbers \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) are called the generalized spectral data of \( L \).

Consider the following inverse problem.

**Inverse Problem 1.** Given the generalized spectral data \( \{\lambda_n, \alpha_n\}_{n \geq 1} \), find \( q(x) \).

Let the functions \( C(x, \lambda), \Phi(x, \lambda) \) be solutions of equation (1) under the conditions

\[
C(0, \lambda) = \Phi(0, \lambda) = 1, \quad C'(0, \lambda) = \Phi(T, \lambda) = 0.
\]

The functions \( \Phi(x, \lambda) \) and \( M(\lambda) := \Phi'(0, \lambda) \) are called the Weyl solution and the Weyl function for \( L \) respectively. According to (6) we have

\[
\Phi(x, \lambda) = \frac{\psi(x, \lambda)}{\Delta(\lambda)} = C(x, \lambda) + M(\lambda)S(x, \lambda),
\]

\( \text{(12)} \)

\[
\langle S(x, \lambda), \Phi(x, \lambda) \rangle \equiv -1,
\]

\( \text{(13)} \)

\[
M(\lambda) = -\frac{d(\lambda)}{\Delta(\lambda)}, \quad d(\lambda) := -\psi'(0, \lambda) = C(T, \lambda).
\]

\( \text{(14)} \)

The function \( d(\lambda) \) is the characteristic function of the boundary value problem for the equation (1) with the boundary conditions \( y'(0) = y(T) = 0 \). Let \( \{\mu_n\}_{n \geq 0} \) be its spectrum. Clearly, \( \{\lambda_n\}_{n \geq 1} \cap \{\mu_n\}_{n \geq 0} = \emptyset \). Thus, \( M(\lambda) \) is a meromorphic function with poles in \( \lambda_n \) and zeros in \( \mu_k \). Moreover, (see, e.g., [6])

\[
\Delta(\lambda) = \frac{T^3}{\pi^2} \prod_{n=1}^{\infty} \frac{\lambda_n - \lambda}{n^2}, \quad d(\lambda) = \frac{T^2}{\pi^2} \prod_{n=0}^{\infty} \frac{\mu_n - \lambda}{(n + 1/2)^2}.
\]

\( \text{(15)} \)

Let \( \lambda = \rho^2 \) and put \( \tau = \text{Im} \rho \). Using the known method (see, e.g., [4]) one can prove the following asymptotics.

**Lemma 1.** (i) For \( |\rho| \to \infty \) the following asymptotics holds

\[
S(x, \lambda) = \frac{\sin \rho x}{\rho} - Q(x) \frac{\cos \rho x}{\rho^2} + \frac{1}{2\rho^2} \int_0^x q(t) \cos \rho (x - 2t) \, dt + O \left( \frac{1}{\rho^2} \exp(|\tau|x) \right),
\]

\( \text{(16)} \)

\[
S'(x, \lambda) = \cos \rho x + Q(x) \frac{\sin \rho x}{\rho} - \frac{1}{2\rho} \int_0^x q(t) \sin \rho (x - 2t) \, dt + O \left( \frac{1}{\rho} \exp(|\tau|x) \right),
\]

\( \text{(17)} \)

\[
\psi(x, \lambda) = \frac{\sin \rho (T - x)}{\rho} - (Q(T) - Q(x)) \frac{\cos \rho(T - x)}{\rho^2} + o \left( \frac{1}{\rho^2} \exp(|\tau|(T - x)) \right),
\]

\[
\psi'(x, \lambda) = -\cos \rho (T - x) - (Q(T) - Q(x)) \frac{\sin \rho(T - x)}{\rho} + o \left( \frac{1}{\rho} \exp(|\tau|(T - x)) \right)
\]

uniformly with respect to \( x \in [0, T] \).
(ii) Fix $\delta > 0$. Then for sufficiently large $|\lambda|$
\[
|\Delta(\lambda)| \geq \frac{C_{\delta}}{|\rho|} \exp(|\tau|T), \quad \lambda \in G_{\delta},
\]
(18)
where $G_{\delta} = \{\lambda = \rho^2 : |\rho - \pi k/T| \geq \delta, k \in \mathbb{Z}\}$.

Using (7), (10), (11), (16) one can calculate
\[
\alpha_n = \frac{T^3}{2\pi^2 n^2} \left(1 + \frac{\kappa_n}{n}\right), \quad \kappa_n = o(1), \quad n \to \infty.
\]
(19)

Fix $k \in S$. According to (14) the function $M(\lambda)$ has a representation
\[
M(\lambda) = \sum_{\nu=0}^{m_k-1} \frac{M_{k+\nu}}{(\lambda - \lambda_k)^{\nu+1}} + M^0_k(\lambda),
\]
(20)
where $M_{k+m_k-1} \neq 0$ and the function $M^0_k(\lambda)$ is regular in a vicinity of $\lambda_k$. The sequence $\{M_n\}_{n \geq 1}$ is called the Weyl sequence for $L$. By virtue of (14), (17), (18) the following estimate holds
\[
M(\lambda) = O(\rho), \quad |\lambda| \to \infty, \quad \lambda \in G_{\delta}.
\]
(21)

Moreover, according to (6), (14), (16), (17) for each fixed $\delta > 0$ we have
\[
M(\rho^2) = i\rho + o(1), \quad |\rho| \to \infty, \quad \arg \rho \in [\delta, \pi - \delta].
\]
(22)

The maximum modulus principle together with (7), (21) give
\[
|M_n| \leq Cn^2.
\]
(23)

Choose $\omega > 0$ such that $\Delta(\pm i\omega) \neq 0$ and put
\[
\beta(\lambda) := \frac{\lambda}{\lambda^2 + \omega^2}, \quad b_{n+\nu} := \frac{1}{\nu!} \beta^{(\nu)}(\lambda_n), \quad n \in \mathbb{S}, \quad \nu = 0, m_n - 1.
\]

According to (23) we get
\[
\left(\frac{1}{\lambda - \lambda_n} + b_n\right)M_n = O\left(\frac{1}{n^2}\right), \quad n \to \infty,
\]
and hence the series
\[
N(\lambda) := \sum_{n \in \mathbb{S}} \sum_{\nu=0}^{m_n-1} \left(\frac{1}{(\lambda - \lambda_n)^{\nu+1}} + b_{n+\nu}\right) M_{n+\nu}.
\]
(24)

converges absolutely and uniformly on compacts that do not include $\lambda_n, n \in \mathbb{S}$.

**Theorem 1.** The following representation holds
\[
M(\lambda) = N(\lambda) + a, \quad a = \lim_{\rho \to +\infty} (i\rho - N(\rho^2)).
\]
(25)

**Proof.** Consider the contour integral
\[
I_N(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_N} \left(\frac{1}{\lambda - \mu} + \beta(\mu)\right) M(\mu) d\mu, \quad \lambda \in \text{int} \Gamma_N,
\]
where the contour $\Gamma_N := \{ \mu : |\mu| = (\pi(N + 1/2)/T)^2 \}$, $N \in \mathbb{N}$, has the counterclockwise circuit. According to (7) we have $\Gamma_N \subset G_\delta$ for sufficiently large $N$ and sufficiently small fixed $\delta > 0$. By virtue of (21) we obtain the estimate

$$
\left( \frac{1}{\lambda - \mu} + \beta(\mu) \right) M(\mu) = O(\mu^{-2})
$$

uniformly with respect to $\lambda$ in bounded subsets of $\mathbb{C}$, and hence

$$
\lim_{N \to \infty} I_N(\lambda) = 0. \tag{26}
$$

On the other hand, using the residue theorem [11] we calculate

$$
I_N(\lambda) = -M(\lambda) + \sum_{n \in S, \lambda_n \in \text{int}\Gamma_N} \left( \text{Res}_{\mu=\lambda_n} \frac{M(\mu)}{\lambda - \mu} + \text{Res}_{\mu=\lambda_n} (M(\mu)\beta(\mu)) \right) + b, \quad \lambda \in \text{int}\Gamma_N \setminus \{ \lambda_n \}_{n \geq 1},
$$

where

$$
b = \text{Res}_{\mu=i\omega} (M(\mu)\beta(\mu)) + \text{Res}_{\mu=-i\omega} (M(\mu)\beta(\mu)) = \frac{M(i\omega) + M(-i\omega)}{2}.
$$

Further we calculate

$$
\text{Res}_{\mu=\lambda_n} \frac{M(\mu)}{\lambda - \mu} = \sum_{\nu=0}^{m_n-1} \frac{M_{n+\nu}}{(\lambda - \lambda_n)^{\nu+1}}, \quad \text{Res}_{\mu=\lambda_n} (M(\mu)\beta(\mu)) = \sum_{\nu=0}^{m_n-1} b_{n+\nu} M_{n+\nu}.
$$

Substituting this into (27) and using (26) we obtain $M(\lambda) = N(\lambda) + b$. By virtue of (22) we get $b = a$ and arrive at (25).

**Theorem 2.** The coefficients $M_n$ and the generalized weight numbers $\alpha_n$ determine each other uniquely by the formula

$$
\sum_{j=0}^{\nu} \alpha_{n+\nu-j} M_{n+m_n-j-1} = -\delta_{\nu,0}, \quad n \in S, \quad \nu = 0, m_n - 1. \tag{28}
$$

**Proof.** Using (10), (14) one can calculate

$$
\sum_{j=0}^{\nu} M_{n+m_n-j-1} \Delta_{n+m_n-j,n} = \psi_n'(0), \quad n \in S, \quad \nu = 0, m_n - 1, \tag{29}
$$

where $\Delta_{p,n} = \Delta^{(p)}(\lambda_n)/(p!)$. Obviously, $\psi_n(x) = \psi_n'(0)S_n(x)$, $n \in S$. Moreover, by virtue of (8), (10) induction gives

$$
\psi_n(x) = \sum_{j=0}^{\nu} \psi_{n+j}'(0)S_{n+\nu-j}(x), \quad n \in S, \quad \nu = 0, m_n - 1. \tag{30}
$$

Further, since

$$
-S''(x, \lambda) + q(x)S(x, \lambda) = \lambda S(x, \lambda), \quad -\psi''(x, \mu) + q(x)\psi(x, \mu) = \mu \psi(x, \mu),
$$

we get

$$
(S(x, \lambda)\psi'(x, \mu) - \psi(x, \mu)S'(x, \lambda))' = (\lambda - \mu)S(x, \lambda)\psi(x, \mu),
$$
and (3), (5), (6) yield
\[ \frac{\Delta(\lambda) - \Delta(\mu)}{\lambda - \mu} = - \int_0^T S(x, \lambda) \psi(x, \mu) \, dx. \]
Hence,
\[ \frac{d}{d\lambda} \Delta(\lambda) = - \int_0^T S(x, \lambda) \psi(x, \lambda) \, dx \]
and we calculate
\[ \Delta_{m_n + \nu,n} = - \frac{1}{m_n + \nu} \sum_{j=0}^{m_n + \nu - 1} \int_0^T \psi_j(x, \lambda_n) S_{m_n + \nu - j}(x, \lambda_n) \, dx, \quad \nu \geq 0. \]

Using (8), (10) and integrating by parts we obtain
\[ \Delta_{m_n + \nu,n} = - \int_0^T \psi_{n+\nu}(x) S_{n+m_n-1}(x) \, dx, \quad n \in \mathbb{S}, \quad \nu = 0, m_n - 1. \quad (31) \]
Substituting (30) in (31) and taking (11) into account, we arrive at
\[ \Delta_{m_n + \nu,n} = - \sum_{j=0}^{\nu} \alpha_{n+\nu-j} \psi_{n+j}'(0), \quad n \in \mathbb{S}, \quad \nu = 0, m_n - 1. \quad (32) \]
Finally, substituting (32) in (29) we get
\[ \sum_{j=0}^{\nu} \psi_{n+\nu-j}'(0) \sum_{k=0}^{j} \alpha_{n+j-k} M_{n+m_n-k-1} = - \psi_{n+\nu}'(0), \quad n \in \mathbb{S}, \quad \nu = 0, m_n - 1. \]
Since \( \psi_{n}'(0) \neq 0, n \in \mathbb{S}, \) by induction we obtain (28).

According to (19) and (28) we have the asymptotics
\[ M_n = - \frac{2\pi^2 n^2}{T^3} \left( 1 + \frac{\kappa_n}{n} \right), \quad \kappa_n = o(1). \quad (33) \]

Consider the following inverse problems.

**Inverse Problem 2.** Given the spectra \( \{\lambda_n\}_{n \geq 1}, \{\mu_n\}_{n \geq 0}, \) construct the function \( q(x). \)

**Inverse Problem 3.** Given the Weyl function \( M(\lambda), \) construct the function \( q(x). \)

**Remark 1.** According to (14), (15), (24), (25), (28) Inverse Problems 1–3 are equivalent. The numbers \( \{\lambda_n, M_n\}_{n \geq 1} \) can also be used as spectral data.

### 3. THE UNIQUENESS THEOREM

We agree that together with \( L \) we consider a boundary value problem \( \tilde{L} = L(\tilde{q}(x), \tilde{T}) \) of the same form but with other potential. If a certain symbol \( \gamma \) denotes an object related to \( L, \) then this symbol with tilde \( \tilde{\gamma} \) denotes the analogous object related to \( \tilde{L} \) and \( \tilde{\gamma} := \gamma - \gamma. \)

**Theorem 3.** If \( \lambda_n = \tilde{\lambda}_n, \alpha_n = \tilde{\alpha}_n, n \geq 1, \) then \( L = \tilde{L}, \) i.e. \( T = \tilde{T}, \) \( q(x) = \tilde{q}(x) \) a.e. on \( (0, T). \) Thus, the specification of the generalized spectral data \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) determines the potential uniquely.
Proof. According to Remark 1 it is sufficient to prove that if \( M(\lambda) = \tilde{M}(\lambda) \), then \( L = \tilde{L} \). Define the matrix \( P(x, \lambda) = [P_{jk}(x, \lambda)]_{j,k=1,2} \) by the formula

\[
P(x, \lambda) \begin{bmatrix} \tilde{S}(x, \lambda) & \tilde{\Phi}(x, \lambda) \\ \tilde{S}'(x, \lambda) & \tilde{\Phi}'(x, \lambda) \end{bmatrix} = \begin{bmatrix} S(x, \lambda) & \Phi(x, \lambda) \\ S'(x, \lambda) & \Phi'(x, \lambda) \end{bmatrix}.
\]

Using (13) and (34) we calculate

\[
P_{11}(x, \lambda) = \Phi^{(j-1)}(x, \lambda)\tilde{S}'(x, \lambda) - S^{(j-1)}(x, \lambda)\tilde{\Phi}(x, \lambda),
\]

\[
P_{12}(x, \lambda) = S^{(j-1)}(x, \lambda)\tilde{\Phi}(x, \lambda) - \Phi^{(j-1)}(x, \lambda)\tilde{S}(x, \lambda),
\]

\[
P_{21}(x, \lambda) = \psi(x, \lambda)\tilde{\Phi}(x, \lambda) - \phi(x, \lambda)\tilde{S}(x, \lambda),
\]

\[
P_{22}(x, \lambda) = 1 + \frac{1}{\Delta(\lambda)} \left( S'(x, \lambda)(\tilde{\psi}(x, \lambda) - \tilde{\psi}'(x, \lambda)) - \psi'(x, \lambda)(\tilde{\psi}(x, \lambda) - \tilde{\psi}'(x, \lambda)) \right).
\]

By virtue of (16)–(18) this yields

\[
P_{11}(x, \lambda) = 1 + O\left(\frac{1}{\rho}\right), \quad P_{12}(x, \lambda) = O\left(\frac{1}{\rho}\right), \quad |\lambda| \to \infty, \quad \lambda \in G_{\delta},
\]

\[
P_{22}(x, \lambda) = 1 + O\left(\frac{1}{\rho}\right), \quad P_{21}(x, \lambda) = O(1), \quad |\lambda| \to \infty, \quad \lambda \in G_{\delta},
\]

uniformly with respect to \( x \in [0, T] \). On the other hand, according to (12) and (35) we get

\[
P_{11}(x, \lambda) = C(x, \lambda)\tilde{S}'(x, \lambda) - S(x, \lambda)\tilde{C}'(x, \lambda) + \hat{M}(\lambda)S(x, \lambda)\tilde{S}'(x, \lambda),
\]

\[
P_{12}(x, \lambda) = S(x, \lambda)\tilde{C}(x, \lambda) - \tilde{S}(x, \lambda)C(x, \lambda) + \hat{M}(\lambda)S(x, \lambda)\tilde{S}(x, \lambda).
\]

Thus, if \( \hat{M}(\lambda) \equiv 0 \), then for each fixed \( x \), the functions \( P_{11}(x, \lambda) \) and \( P_{12}(x, \lambda) \) are entire in \( \lambda \). Together with (37) this yields \( P_{11}(x, \lambda) \equiv 1, P_{12}(x, \lambda) \equiv 0 \). Substituting into (36) we get \( S(x, \lambda) \equiv \tilde{S}(x, \lambda) \) for all \( x \) and \( \lambda \) and consequently \( L = \tilde{L} \). \( \square \)

4. MAIN EQUATION. SOLUTION OF THE INVERSE PROBLEM

Let the spectral data \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) of \( L = L(q(x), T) \) be given. We choose an arbitrary model boundary value problem \( \tilde{L} = L(q(x), T) \) (e.g., one can take \( \tilde{q}(x) \equiv 0 \)). Introduce the numbers \( \xi_n, n \geq 1 \), by the formulae

\[
\xi_{k+\nu} := |\rho_k - \tilde{\rho}_k| + \frac{1}{k^2} \sum_{p=\nu}^{m_k-1} \left| M_{k+p} - \tilde{M}_{k+p} \right|, \quad k \in \mathbb{S} \cap \mathbb{S}, \quad m_k = \tilde{m}_k, \quad \nu = 0, m_k - 1,
\]

\[
\xi_n := 1 \text{ for the rest of } n.
\]
According to (7) and (33) we have

\[ \xi_n = O\left(\frac{1}{n}\right), \quad n \to \infty. \]  

(40)

Denote

\[ \lambda_{n,0} := \lambda_n, \quad \lambda_{n,1} := \tilde{\lambda}_n, \quad M_{n,0} := M_n, \quad M_{n,1} := \tilde{M}_n, \]

\[ S_0 := S, \quad S_1 := \tilde{S}, \quad m_{n,0} := m_n, \quad m_{n,1} := \tilde{m}_n, \]

\[ S_{n+i,0}(x) := S_{\nu}(x, \lambda_{n,i}), \quad \tilde{S}_{n+i,0}(x) := \tilde{S}_{\nu}(x, \lambda_{n,i}), \quad n \in S_0, \quad \nu = 0, m_{n,1} - 1, \quad i = 0, 1, \]

\[ D(x, \lambda, \mu) := \frac{\langle S(x, \lambda), S(x, \mu) \rangle}{\lambda - \mu} = \int_0^x S(t, \lambda), S(t, \mu) \, dt, \]

\[ D_{\nu,0}(x, \lambda, \mu) := \frac{1}{\nu!} \frac{\partial^{\nu+n}}{\partial \lambda^n \mu^\nu} D(x, \lambda, \mu). \]

For \( i, j = 0, 1, \quad n \in S_i \) put

\[ A_{n+i,0}(x, \lambda) := \sum_{p=0}^{m_{n,i}-1} M_{n+p,i} D_{0,p-\nu}(x, \lambda, \lambda_{n,i}), \quad P_{n+i,k,j}(x) = \frac{1}{\nu!} \frac{\partial^{\nu}}{\partial \lambda^{k}} A_{k,j}(x, \lambda) \bigg|_{\lambda=\lambda_{n,i}}, \]

where \( k \geq 1, \quad \nu = 0, m_{n,1} - 1. \) Analogously we define \( \tilde{D}(x, \lambda, \mu), \quad \tilde{D}_{\nu,0}(x, \lambda, \mu), \quad \tilde{A}_{n,i}(x, \lambda) \)
and \( \tilde{P}_{n,i,k,j}(x), \quad n, k \geq 1, \quad i, j = 0, 1, \quad \nu = 0, 1 : \)

\[ |S_{n,0}(x)| \leq Cn^{\nu-1}, \quad |S_{n,0}(x) - S_{n,1}(x)| \leq C\xi_n n^{\nu-1}, \]

(41)

\[ D(x, \lambda, \lambda_{k,j})| \leq \frac{C \exp(|\tau| \frac{\lambda}{k})}{|\rho k|(|\rho + \pi k/T| + 1)}, \]

\[ D(x, \lambda, \lambda_{k,0}) - D(x, \lambda, \lambda_{k,1})| \leq \frac{C \xi_k \exp(|\tau| \frac{\lambda}{k})}{|\rho k|(|\rho + \pi k/T| + 1)}, \]

(42)

\[ |P_{n,i,k,j}(x)| \leq \frac{C k}{(n-k+1)n}, \quad |P_{n,i,k,j}(x)| \leq C\left(\nu k + \frac{k^{\nu+1}}{n}\right), \]

(43)

The analogous estimates are also valid for \( \tilde{S}_{n,i}(x), \quad \tilde{D}(x, \lambda, \lambda_{k,j}), \quad \tilde{P}_{n,i,k,j}(x). \)
Lemma 2. The following relation holds

$$\tilde{S}_{n,i}(x) = S_{n,i}(x) - \sum_{k=1}^{\infty} \left( \tilde{P}_{n,i,k,0}(x)S_{k,0}(x) - \tilde{P}_{n,i,k,1}(x)S_{k,1}(x) \right), \quad n \geq 1, \ i = 0,1, \quad (44)$$

where the series converges absolutely and uniformly with respect to $x \in [0,T]$.

Proof. Let real numbers $a$, $b$ be such that $a < \min \Re \lambda_{n,i}$, $b > \max |\Im \lambda_{n,i}|$, $n \geq 1$, $i = 0,1$. In the $\lambda$-plane consider closed contour $\gamma_N := \partial \Omega_N$ (with counterclockwise circuit), where $\Omega_N = \{ \lambda : a \leq \Re \lambda \leq (N + 1/2)^2 \pi^2/T^2, |\Im \lambda| \leq b \}$. By the standard method (see [6]), using (12), (35)–(37) and Cauchy’s integral formula [11], we obtain the representation

$$\tilde{S}(x, \lambda) = S(x, \lambda) - \frac{1}{2\pi i} \int_{\gamma_N} \tilde{M}(\mu) \tilde{D}(x, \lambda, \mu) S(x, \mu) d\mu + \varepsilon_N(x, \lambda), \quad (45)$$

where

$$\lim_{N \to \infty} \frac{\partial^{\nu}}{\partial \lambda^{\nu}} \varepsilon_N(x, \lambda) = 0, \ \nu \geq 0,$$

uniformly with respect to $x \in [0,T]$ and $\lambda$ on bounded sets. Calculating the integral in (45) by the residue theorem and using (20) we get

$$\frac{1}{2\pi i} \int_{\gamma_N} \tilde{M}(\mu) \tilde{D}(x, \lambda, \mu) S(x, \mu) d\mu = \sum_{k=1}^{N} \left( \tilde{A}_{k,0}(x, \lambda)S_{k,0}(x) - \tilde{A}_{k,1}(x, \lambda)S_{k,1}(x) \right)$$

for sufficiently large $N$. Passing to the limit in (45) as $N \to \infty$ we obtain

$$\tilde{S}(x, \lambda) = S(x, \lambda) - \sum_{k=1}^{\infty} \left( \tilde{A}_{k,0}(x, \lambda)S_{k,0}(x) - \tilde{A}_{k,1}(x, \lambda)S_{k,1}(x) \right). \quad (46)$$

Differentiating this with respect to $\lambda$ the corresponding number of times and then taking $\lambda = \lambda_{n,i}$, we arrive at (44).

Analogously to (46) one can obtain the following relation

$$\tilde{\Phi}(x, \lambda) = \Phi(x, \lambda) - \sum_{k=1}^{\infty} \left( \tilde{F}_{k,0}(x, \lambda)S_{k,0}(x) - \tilde{F}_{k,1}(x, \lambda)S_{k,1}(x) \right), \quad (47)$$

where

$$\tilde{F}_{n+\nu,i}(x, \lambda) := \sum_{\nu=\nu}^{m_{n,i}-1} M_{n+p,i} \tilde{G}_{\nu}(x, \lambda, \lambda_{n,i}), \quad n \in \mathbb{S}_i, \ \nu = 0, m_{n,i} - 1, \ i = 0,1,$$

$$\tilde{G}_{\nu}(x, \lambda, \mu) = \frac{1}{\nu!} \frac{d^{\nu}}{d\mu^{\nu}} \tilde{G}(x, \lambda, \mu),$$

$$\tilde{G}(x, \lambda, \mu) := \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{S}(x, \mu) \rangle}{\lambda - \mu} = \frac{1}{\lambda - \mu} + \int_{0}^{x} \tilde{\Phi}(t, \lambda) \tilde{S}(t, \mu) dt.$$

For each fixed $x \in [0,T]$ the relation (44) can be considered as a system of linear equations with respect to $S_{n,i}(x)$, $n \geq 1$, $i = 0,1$. But the series in (44) converges only "with brackets", i.e. the terms in them cannot be dissociated. Therefore, it is inconvenient to
use (44) as a main equation of the inverse problem. Below we will transfer (44) to a linear equation in the Banach space of bounded sequences (see (53)).

Let \( w \) be the set of indices \( u = (n, i) \), \( n \geq 1 \), \( i = 0, 1 \). For each fixed \( x \in [0, T] \) we define the vector

\[
\phi(x) = [\phi_u(x)]^T_{u \in w} = [S_{n,0}(x), S_{n,1}(x)]^T_{n \geq 1}
\]

(where \( T \) is the sign for transposition) by the formula

\[
\begin{bmatrix}
\phi_{n,0}(x) \\
\phi_{n,1}(x)
\end{bmatrix} = n \begin{bmatrix}
\chi_n & -\chi_n \\
0 & 1
\end{bmatrix} \begin{bmatrix}
S_{n,0}(x) \\
S_{n,1}(x)
\end{bmatrix}, \quad \chi_n = \begin{cases}
\xi_n^{-1}, & \xi_n \neq 0, \\
0, & \xi_n = 0.
\end{cases}
\]

Note that if \( \phi_{n,0}, \phi_{n,1} \) are given, then \( S_{n,0}, S_{n,1} \) can be found by the formula

\[
\begin{bmatrix}
S_{n,0}(x) \\
S_{n,1}(x)
\end{bmatrix} = \frac{1}{n} \begin{bmatrix}
\xi_n & 1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\phi_{n,0}(x) \\
\phi_{n,1}(x)
\end{bmatrix}.
\]

(48)

Consider also a block-matrix

\[
H(x) = [H_{u,v}(x)]_{u,v \in w} = \begin{bmatrix}
H_{n,0;0,0}(x) & H_{n,0;0,1}(x) \\
H_{n,1;0,0}(x) & H_{n,1;0,1}(x)
\end{bmatrix}_{n,k \geq 1}, \quad u = (n, i), \ v = (k, j),
\]

where

\[
\begin{bmatrix}
H_{n,0;0,0}(x) & H_{n,0;0,1}(x) \\
H_{n,1;0,0}(x) & H_{n,1;0,1}(x)
\end{bmatrix} = n \begin{bmatrix}
\chi_n & -\chi_n \\
0 & 1
\end{bmatrix} \begin{bmatrix}
P_{n,0;0,0}(x) & P_{n,0;0,1}(x) \\
P_{n,1;0,0}(x) & P_{n,1;0,1}(x)
\end{bmatrix} \begin{bmatrix}
\xi_k & 1 \\
0 & -1
\end{bmatrix}.
\]

Analogously we introduce \( \tilde{\phi}_{n,i}'(x) \), \( \tilde{\phi}(x) \) and \( \tilde{H}_{n,i;k,j}(x) \), \( \tilde{H}(x) \) by the replacement of \( S_{n,i}(x), \ P_{n,i;k,j}(x) \) in the preceding definitions with \( S_{n,i}(x), \ P_{n,i;k,j}(x) \) respectively. Using (41), (43) we get the estimates

\[
|\phi_{n,i}'(x)| \leq Cn^\nu, \quad |H_{n,i;k,j}(x)| \leq \frac{C\xi_k}{|n-k| + 1}, \quad |H_{n,i;k,j}(x)| \leq C\xi_k(n + k)^\nu, \quad \nu = 0, 1, \quad (49)
\]

\[
|\tilde{\phi}_{n,i}'(x)| \leq Cn^\nu, \quad |\tilde{H}_{n,i;k,j}(x)| \leq \frac{C\xi_k}{|n-k| + 1}, \quad |\tilde{H}_{n,i;k,j}(x)| \leq C\xi_k(n + k)^\nu, \quad \nu = 0, 1, \quad (50)
\]

\[
|\tilde{H}_{n,i;k,j}(x) - \tilde{H}_{n,i;k,j}(x_0)| \leq C\xi_k|x - x_0|, \quad x, x_0 \in [0, T]. \quad (51)
\]

Consider the Banach space \( B \) of bounded sequences \( a = [a_u]_{u \in w} \) with the norm \( ||a||_B = \sup_{u \in w} |a_u| \). It follows from (49), (50) that for each fixed \( x \in [0, T] \) the operators \( H(x) \) and \( \tilde{H}(x) \), acting from \( B \) to \( B \), are linear bounded ones, and

\[
||H(x)||_{B \to B}, \ ||\tilde{H}(x)||_{B \to B} \leq C \sup_{n \geq 1} \sum_{k=1}^{\infty} \frac{\xi_k}{|n-k| + 1} < \infty. \quad (52)
\]

**Theorem 4.** For each fixed \( x \in [0, T] \) the vector \( \phi(x) \in B \) satisfies the equation

\[
\tilde{\phi}(x) = (I - \tilde{H}(x))\phi(x) \quad (53)
\]

in the Banach space \( B \), where \( I \) is the identity operator.

**Proof.** We rewrite (44) in the form

\[
\begin{bmatrix}
\tilde{S}_{n,0}(x) \\
\tilde{S}_{n,1}(x)
\end{bmatrix} = \begin{bmatrix}
S_{n,0}(x) \\
S_{n,1}(x)
\end{bmatrix} - \sum_{k=1}^{\infty} \begin{bmatrix}
\tilde{P}_{n,0;0,0}(x) & -\tilde{P}_{n,0;0,1}(x) \\
\tilde{P}_{n,1;0,0}(x) & -\tilde{P}_{n,1;0,1}(x)
\end{bmatrix} \begin{bmatrix}
S_{k,0}(x) \\
S_{k,1}(x)
\end{bmatrix}, \quad n \geq 1.
\]
Substituting here (48) and taking into account our notations we arrive at
\[ \hat{\phi}_{n,i}(x) = \phi_{n,i}(x) - \sum_{(k,j) \in w} \hat{H}_{n,i;k,j}(x)\phi_{k,j}(x), \quad (n,i) \in w, \]
which is equivalent to (53).

For each fixed \( x \in [0,T] \) the relation (53) can be considered as a linear equation with respect to \( \phi(x) \). This equation is called the main equation of the inverse problem. Thus, the nonlinear inverse problem is reduced to the solution of the linear equation. Let us prove the unique solvability of the main equation.

**Theorem 5.** For each fixed \( x \in [0,T] \) the operator \( I - \tilde{H}(x) \) has a bounded inverse operator, namely \( I + H(x) \), i.e. the main equation (53) is uniquely solvable.

**Proof.** Acting in the same way as in Lemma 2 and using (38), we obtain
\[ D(x,\lambda,\mu) - \tilde{D}(x,\lambda,\mu) = \frac{1}{2\pi i} \int_{\gamma_N} \tilde{D}(x,\lambda,\xi)\tilde{M}(\xi)D(x,\xi,\mu)d\xi + \varepsilon_1(x,\lambda,\mu), \]
where
\[ \lim_{N \to \infty} \frac{\partial^{\nu+j}}{\partial \lambda^\nu \partial \mu^j} \varepsilon_1(x,\lambda,\mu) = 0, \quad \nu, j \geq 0, \]
uniformly with respect to \( x \in [0,T] \) and \( \lambda, \mu \) on bounded sets. Calculating the integral by the residue theorem and passing to the limit as \( N \to \infty \) we obtain
\[ D(x,\lambda,\mu) - \tilde{D}(x,\lambda,\mu) = \sum_{p=0}^{1} (-1)^p \sum_{l \in S_p} \sum_{\nu=0}^{m_{l,p}-1} D_{\nu,0}(x,\lambda_{l,p},\mu)\tilde{A}_{\nu+p}(x,\lambda). \]

According to the definition of \( P_{n,i;k,j}(x), \tilde{P}_{n,i;k,j}(x) \) we arrive at
\[ P_{n,i;k,j}(x) - \tilde{P}_{n,i;k,j}(x) = \sum_{l=0}^{\infty} \left( \tilde{P}_{n,i;l,0}(x)P_{l,0;k,j}(x) - \tilde{P}_{n,i;l,1}(x)P_{l,1;k,j}(x) \right), \quad n, k \geq 1, \ i, j = 0, 1. \]

Further, taking the definition of \( H_{n,i;k,j}(x), \tilde{H}_{n,i;k,j}(x) \) into account we get
\[ H_{n,i;k,j}(x) - \tilde{H}_{n,i;k,j}(x) = \sum_{(l,p) \in w} \tilde{H}_{n,i;l,p}(x)H_{l,p;k,j}(x), \quad (n,i), (k,j) \in w, \]
which is equivalent to \( (I - \tilde{H}(x))(I + H(x)) = I \). Symmetrically one gets
\[ (I + H(x))(I - \tilde{H}(x)) = I. \]

Hence the operator \( (I - \tilde{H}(x))^{-1} \) exists, and it is a linear bounded operator.

Using the solution of the main equation one can construct the function \( q(x) \). Thus, we obtain the following algorithm for solving the inverse problem.

**Algorithm 1.** Let the spectral data \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) be given. Then
\((i)\) construct \( M_n, n \geq 1, \) by solving the linear systems (28);
\((ii)\) choose \( \tilde{L} \) and calculate \( \tilde{\phi}(x) \) and \( \tilde{H}(x) \);
(iii) find \( \phi(x) \) by solving equation (53);
(iv) choose \( n \in \mathbb{S} \) (e.g., \( n = 1 \)) and construct \( q(x) \) by the formula

\[
q(x) = \frac{\phi''_{n,1}(x)}{\phi'_{n,1}(x)} + \lambda_n.
\]

**Remark 2.** In the particular case when \( \lambda_n = \tilde{\lambda}_n, \alpha_n = \tilde{\alpha}_n \) for \( n > N \) (let for definiteness \( N + 1 \in \mathbb{S} \cap \tilde{\mathbb{S}} \)) according to (44) and the definition of \( S_{n,i}(x), \tilde{P}_{n,i,k,j}(x) \) the main equation becomes a linear algebraic system

\[
\tilde{S}_{n,i}(x) = S_{n,i}(x) - \sum_{k=1}^{N} \left( \tilde{P}_{n,i,k,0}(x)S_{k,0}(x) - \tilde{P}_{n,i,k,1}(x)S_{k,1}(x) \right), \quad n = 1, N, \ i = 0, 1,
\]

whose determinant does not vanish for any \( x \in [0, T] \) by virtue of Theorem 5.

In the next section for the case \( q(x) \in L_2(0, T) \) we give another algorithm, which is used in Section 6 for obtaining necessary and sufficient conditions for the solvability of the inverse problem.

**5. ALGORITHM 2**

Here and in the sequel we assume that \( q(x) \in L_2(0, T) \). It is known that then \( \{\kappa_n\} \in l_2 \) in formulae (7), (19), (33). We agree that below one and the same symbol \( \{\kappa_n\} \) denotes different sequences in \( l_2 \). Let us choose the model boundary value problem \( L = L(\bar{q}(x), T) \) so that \( \omega = \tilde{\omega} \) (for example, one can take \( \bar{q}(x) \equiv 2\omega/T \)). Then besides (40) according to (7), (33), (39) we have

\[
\xi_n = \frac{\kappa_n}{n}, \quad \Omega := \left( \frac{1}{\sum_{n=1}^{\infty} (n\xi_n)^2} \right)^\frac{1}{2} < \infty, \quad \sum_{n=1}^{\infty} \xi_n < \infty.
\]

Denote

\[
\tilde{B}_{n+i}(x) := \sum_{\nu=0}^{m_{n,i}-1} M_{n+\nu,i}\tilde{S}_{n+\nu,i}(x), \quad n \in S_i, \quad \nu = 0, m_{n,i} - 1, \ i = 0, 1,
\]

\[
\varepsilon_0(x) := \sum_{k=1}^{\infty} \left( \tilde{B}_{k,0}(x)S_{k,0}(x) - \tilde{B}_{k,1}(x)S_{k,1}(x) \right), \quad \varepsilon(x) := 2\varepsilon_0(x).
\]

It is obvious that

\[
\tilde{A}_{n,i}'(x, \lambda) = \tilde{S}(x, \lambda)\tilde{B}_{n,i}(x), \quad n \geq 1, \quad i = 0, 1.
\]

**Lemma 3.** The series in (57) converges absolutely and uniformly on \( [0, T] \) and allows termwise differentiation. The function \( \varepsilon_0(x) \) is absolutely continuous, and \( \varepsilon(x) \in L_2(0, T) \).

**Proof.** It is sufficient to prove for the case \( m_{n,i} = 1, n \geq 1, i = 0, 1 \). We rewrite \( \varepsilon_0(x) \) to the form \( \varepsilon_0(x) = A_1(x) + A_2(x) \), where

\[
A_1(x) = \sum_{k=1}^{\infty} (M_{k,0} - M_{k,1})\tilde{S}_{k,0}(x)S_{k,0}(x),
\]

\[
A_2(x) = \sum_{k=1}^{\infty} M_{k,1} \left( (\tilde{S}_{k,0}(x) - \tilde{S}_{k,1}(x))S_{k,0}(x) + \tilde{S}_{k,1}(x)(S_{k,0}(x) - S_{k,1}(x)) \right).
\]
It follows from (33), (41) and (55) that the series in (58) converge absolutely and uniformly on \([0, T]\), and

\[ |A_j(x)| \leq C \sum_{k=0}^{\infty} \xi_k \leq C\Omega, \quad j = 1, 2. \]

Furthermore, using the asymptotic formulae (7), (16) and (33) we calculate

\[
A_1'(x) = \sum_{k=1}^{\infty} (M_{k,0} - M_{k,1}) \frac{d}{dx} \left( S_{k,0}(x)S_{k,0}(x) \right) = \sum_{k=1}^{\infty} \kappa_k \left( \sin 2kx + O\left( \frac{1}{k} \right) \right).
\]

Hence \(A_1(x) \in W_2^1[0, T]\). Similarly, we get \(A_2(x) \in W_2^1[0, T]\), and consequently \(\varepsilon_0(x) \in W_2^1[0, T]\).

**Lemma 4.** The following relations hold

\[ q(x) = \tilde{q}(x) + \varepsilon(x). \]  

**Proof.** Differentiating (46) twice with respect to \(x\) and using (57) we get

\[
\tilde{S}'(x, \lambda) = S'(x, \lambda) - \varepsilon_0(x)\tilde{S}(x, \lambda) - \sum_{k=1}^{\infty} \left( \tilde{A}_{k,0}(x, \lambda)S_{k,0}'(x) - \tilde{A}_{k,1}(x, \lambda)S_{k,1}'(x) \right).
\]

\[
\tilde{S}''(x, \lambda) = S''(x, \lambda) - \sum_{k=1}^{\infty} \left( (\tilde{S}(x, \lambda)\tilde{B}_{k,0}(x))'S_{k,0}(x) - (\tilde{S}(x, \lambda)\tilde{B}_{k,1}(x))'S_{k,1}(x) \right) - 2\tilde{S}(x, \lambda) \sum_{k=1}^{\infty} \left( \tilde{B}_{k,0}(x)S_{k,0}'(x) - \tilde{B}_{k,1}(x)S_{k,1}'(x) \right) - \sum_{k=1}^{\infty} \left( \tilde{A}_{k,0}(x, \lambda)S_{k,0}''(x) - \tilde{A}_{k,1}(x, \lambda)S_{k,1}''(x) \right).
\]

Using (1), (8) we replace here the second derivatives and then replace \(S(x, \lambda)\) using (46). This yields

\[
\tilde{q}(x)\tilde{S}(x, \lambda) = 2\tilde{S}(x, \lambda) \sum_{k=1}^{\infty} \left( \tilde{B}_{k,0}(x)S_{k,0}'(x) - \tilde{B}_{k,1}(x)S_{k,1}'(x) \right) +
\]

\[ + \sum_{k=1}^{\infty} \left( (\tilde{S}(x, \lambda)\tilde{B}_{k,0}(x))'S_{k,0}(x) - (\tilde{S}(x, \lambda)\tilde{B}_{k,1}(x))'S_{k,1}(x) \right) +
\]

\[ + \sum_{k=1}^{\infty} \left( (\lambda - \lambda_{k,0})\tilde{A}_{k,0}(x, \lambda)S_{k,0}(x) - (\lambda - \lambda_{k,1})\tilde{A}_{k,1}(x, \lambda)S_{k,1}(x) \right) - \mathcal{A}(x, \lambda), \]  

where

\[
\mathcal{A}(x, \lambda) = \sum_{j=0}^{1} (-1)^j \sum_{k \in \mathbb{S}_j \atop m_{k,j} \geq 2} \sum_{\nu=0}^{m_{k,j}-2} \tilde{A}_{k+\nu+1,j}(x, \lambda)S_{k+\nu,j}(x).
\]

Using (1), (8) for \(j = 0, 1, \quad k \in \mathbb{S}_j \), \(\nu = 0, m_{k,j} - 1\) we calculate

\[
(\tilde{S}(x, \lambda)\tilde{B}_{k+\nu,j}(x))' + (\lambda - \lambda_{k,j})\tilde{A}_{k+\nu,j}(x, \lambda) = 2\tilde{S}(x, \lambda)\tilde{B}_{k+\nu,j}'(x) + (1 - \delta_{\nu, m_{k,j} - 1})\tilde{A}_{k+\nu+1,j}(x, \lambda).
\]

Applying this relation we get

\[ \sum_{k=1}^{\infty} \left( (\tilde{S}(x, \lambda)\tilde{B}_{k,0}(x))'S_{k,0}(x) - (\tilde{S}(x, \lambda)\tilde{B}_{k,1}(x))'S_{k,1}(x) \right) +
\]
\[ + \sum_{k=1}^{\infty} \left( (\lambda - \lambda_{k,0})\tilde{A}_{k,0}(x, \lambda)S_{k,0}(x) - (\lambda - \lambda_{k,1})\tilde{A}_{k,1}(x, \lambda)S_{k,1}(x) \right) = \]

\[ = 2\tilde{S}(x, \lambda) \sum_{k=1}^{\infty} \left( \tilde{B}_{k,0}'(x)S_{k,0}(x) - \tilde{B}_{k,1}'(x)S_{k,1}(x) \right) + A(x, \lambda), \]

which together with (57), (60) gives (59).

Thus, we obtain the following algorithm for solving the inverse problem.

**Algorithm 2.** Let the spectral data \( \{\lambda_n, \alpha_n\}_{n \ge 1} \) be given. Then

(i) construct \( M_n, n \ge 1 \), by solving the linear systems (28);

(ii) choose \( \tilde{L} \) so that \( \omega = \tilde{\omega} \) and calculate \( \tilde{\phi}(x) \) and \( \tilde{H}(x) \);

(iii) find \( \phi(x) \) by solving equation (53) and calculate \( S_{n,j}(x), n \ge 1, j = 0, 1, \) by (48);

(iv) calculate \( q(x) \) by formulae (56), (57), (59).

### 6. NECESSARY AND SUFFICIENT CONDITIONS

In the present section we obtain necessary and sufficient conditions for the solvability of the inverse problem. In the general non-selfadjoint case they must include the requirement of the solvability of the main equation. In Section 7 some important cases will be considered when the solvability of the main equation can be proved by sufficiency, namely: the selfadjoint case, the case of finite-dimensional perturbations of the spectral data and the case of small their perturbations.

**Theorem 6.** For complex numbers \( \{\lambda_n, \alpha_n\}_{n \ge 1} \) to be the spectral data of a certain boundary value problem \( L(q(x), T) \) with \( q(x) \in L_2(0, T) \) it is necessary and sufficient that:

(i) the relations (7), (19) hold with \( \{\kappa_n\} \in l_2 \);

(ii) \( \alpha_n \neq 0 \) for all \( n \in \mathbb{S} \);

(iii) (Condition \( S \)) for each \( x \in [0, T] \) the linear bounded operator \( I - \tilde{H}(x) \), acting from \( B \) to \( B \), has a bounded inverse one. Here \( \tilde{L} \) is chosen so that \( \tilde{\omega} = \omega \).

The boundary value problem \( L = L(q(x), T) \) can be constructed by Algorithms 1, 2.

We note that by sufficiency condition (ii) of the theorem allows to solve linear systems (28) for finding \( M_n, n \ge 1 \), which are used for constructing the main equation. Moreover, we have

\[ M_{n+m_n-1} \neq 0, \quad n \in \mathbb{S}. \]  

(61)

Let \( \phi(x) = [\phi_u(x)]_{u \in w} \) be the solution of the main equation (53). Denote

\[ H(x) = [H_{u,v}(x)]_{u,v \in w} := (I - \tilde{H}(x))^{-1} - I, \]

i.e.

\[ (I - \tilde{H}(x))(I + H(x)) = (I + H(x))(I - \tilde{H}(x)) = I. \]  

(62)

Similarly to Lemma 1.6.7 in [6] using (51), (53) one can prove the following assertion.

**Lemma 5.** For \( n, k \ge 1, i, j, \nu = 0, 1, x \in [0, T] \) the following relations hold

\[ \phi_{n,i}(x) \in C^1[0, T], \quad |\phi_{n,i}^{(\nu)}(x)| \le Cn^\nu, \]

(63)

\[ |\phi_{n,i}^{(\nu)}(x) - \phi_{n,i}^{(\nu)}(x)| \le C \Omega_n^{1-\nu}, \]

(64)

\[ |H_{n,i,n+k,j}(x)| \le C \xi_k \left( \frac{1}{|n-k|+1} + \Omega_n \right), \]

(65)
\[
|H_{n,i;k,j}(x)| \leq C\xi_k \left( \frac{1}{|n-k| + 1} + \Omega \eta_k \right),
\]
where \(\Omega\) is defined in (55), and
\[
\eta_n := \left( \sum_{k=1}^{\infty} \frac{1}{k^2 (|n-k| + 1)^2} \right)^{1/2}.
\]

We define the functions \(S_{n,i}(x)\) by formulae (48) and according to (63) we get (41). Then (44) is also valid. By virtue of (48), (64) and Lemma 5 we have
\[
|S_{n,i}^{(\nu)}(x) - \tilde{S}_{n,i}^{(\nu)}(x)| \leq \frac{C}{n} \Omega \eta_1^{-\nu},
\]
Furthermore, we construct the functions \(S(x, \lambda)\) and \(\Phi(x, \lambda)\) via (46), (47) and the function \(q(x)\) by formulae (56), (57), (59). Clearly,
\[
S_{n,i}(x, \lambda_{n,i}) = S_{n+i,0}(x), \quad n \in S_i, \quad \nu = 0, m_{n,i} - 1, \quad i = 0, 1.
\]
Analogously to Lemma 1.6.8 in [6] using (41), (68) one can prove the following assertion.

**Lemma 6.** \(q(x) \in L_2(0, T)\).

**Lemma 7.** For \(i = 0, 1, \ n \in S_i, \ \nu = \frac{1}{2}, m_{n,i} - 1\) the following relations hold:
\[
\ell S_{n,i}(x) = \lambda_n S_{n,i}(x), \quad \ell S_{n+i,0}(x) = \lambda_n S_{n+i,0}(x) + S_{n+i-1,0}(x),
\]
\[
\ell S(x, \lambda) = \lambda S(x, \lambda), \quad \ell \Phi(x, \lambda) = \lambda \Phi(x, \lambda),
\]
\[
S(0, \lambda) = 0, \quad S'(0, \lambda) = 1, \quad \Phi(0, \lambda) = 1, \quad \Phi(T, \lambda) = 0.
\]

**Proof.** 1) According to the estimates (42) the series in (46) is termwise differentiable with respect to \(x\), and hence \(S(0, \lambda) = 0, \quad S'(0, \lambda) = 1\). By virtue of (69) we have \(S_{n,j}(0) = 0, \quad (n, j) \in w\). Thus, formula (47) gives \(\Phi(0, \lambda) = 1\).

2) In order to prove (70), (71) we first assume that
\[
\Omega_1 := \left( \sum_{k=0}^{\infty} (k^2 \xi_k^2)^{1/2} \right) < \infty.
\]

Differentiating (62) twice we obtain
\[
H''(x) = (I + H(x)) \tilde{H}''(x)(E + H(x)) + H'(x) \tilde{H}'(x)(I + H(x)) + (I + H(x)) \tilde{H}'(x) H(x).
\]
It follows from (50), (65), (66) that the series in (74) converge absolutely and uniformly for \(x \in [0, T]\), \(H_{n,i;k,j}(x) \in C^2[0, T]\), and
\[
|H''_{n,i;k,j}(x)| \leq C\xi_k(n + k).
\]
Solving the main equation (53) we infer
\[
\phi_{n,i}(x) = \tilde{\phi}_{n,i}(x) + \sum_{k,j} H_{n,i;k,j}(x) \tilde{\phi}_{k,j}(x), \quad x \in [0, T], \quad (n, i) , (k, j) \in w.
\]
According to (50) and (65) the series in (76) converges absolutely and uniformly for \( x \in [0, T] \). Further, using (76) we calculate

\[
\tilde{\ell}_n(x) = \tilde{\ell}_n(x) + \sum_{k,j} H_{n,i;k,j}(x)\tilde{\phi}_{k,j}(x) - 2 \sum_{k,j} H_{n,i;k,j}'(x)\tilde{\phi}_{k,j}'(x) - \sum_{k,j} H_{n,i;k,j}''(x)\tilde{\phi}_{k,j}(x),
\]

where according to (50), (66), (67) and (75) the series converge absolutely and uniformly for \( x \in [0, T] \), and

\[
\tilde{\phi}_{n,i}(x) \in C[0, T], \quad |\tilde{\phi}_{n,i}(x)| \leq Cn^2, \quad (n, i) \in w.
\]

On the other hand, it follows from the proof of Lemma 3 and from (73) that \( q(x) - \tilde{q}(x) \in C[0, T] \); hence

\[
\ell_{n,i}(x) \in C[0, T], \quad |\ell_{n,i}(x)| \leq Cn^2, \quad (n, i) \in w.
\]

Together with (48) this implies that

\[
\ell_{S_{n,i}}(x) = C[0, T], \quad |\ell_{S_{n,i}}(x)| \leq Cn, \quad |\ell_{S_{n,0}}(x) - \ell_{S_{n,1}}(x)| \leq Cn\xi_n, \quad (n, i) \in w.
\]

Using (44), (57), (59) we get

\[
\tilde{\ell}_S_{n,i}(x) = \ell_{S_{n,i}}(x) - \sum_{k=1}^{\infty} \left( \tilde{P}_{n,i;k,0}(x)\ell_{S_{n,0}}(x) - \tilde{P}_{n,i;k,1}(x)\ell_{S_{n,1}}(x) \right) -
\]

\[
- \sum_{k=1}^{\infty} \left( \langle \tilde{S}_{n,i}(x), \tilde{B}_{k,0}(x) \rangle S_{k,0}(x) - \langle \tilde{S}_{n,i}(x), \tilde{B}_{k,1}(x) \rangle S_{k,1}(x) \right), \quad (n, i) \in w. \tag{77}
\]

Similarly using (46) and (47) we calculate

\[
\tilde{\ell}_\Phi(x, \lambda) = \ell_\Phi(x, \lambda) - \sum_{k=1}^{\infty} \left( \tilde{\Phi}_{k,0}(x, \lambda)\ell_{S_{k,0}}(x) - \tilde{\Phi}_{k,1}(x, \lambda)\ell_{S_{k,1}}(x) \right) -
\]

\[
- \sum_{k=1}^{\infty} \left( \langle \tilde{\Phi}(x, \lambda), \tilde{B}_{k,0}(x) \rangle S_{k,0}(x) - \langle \tilde{\Phi}(x, \lambda), \tilde{B}_{k,1}(x) \rangle S_{k,1}(x) \right), \quad (n, i) \in w. \tag{78}
\]

\[
\tilde{\Phi}(x, \lambda) = \ell_\Phi(x, \lambda) - \sum_{k=1}^{\infty} \tilde{\Phi}_{k,0}(x, \lambda)\ell_{S_{k,0}}(x) - \tilde{\Phi}_{k,1}(x, \lambda)\ell_{S_{k,1}}(x) - \sum_{k=1}^{\infty} \left( \langle \tilde{\Phi}(x, \lambda), \tilde{B}_{k,0}(x) \rangle S_{k,0}(x) - \langle \tilde{\Phi}(x, \lambda), \tilde{B}_{k,1}(x) \rangle S_{k,1}(x) \right). \tag{79}
\]

For \( n \in \mathbb{S} \), and \( i = 0, 1 \) it follows from (77) that

\[
\lambda_{n,i} \tilde{S}_{n,i}(x) = \ell_{S_{n,i}}(x) - \sum_{k=1}^{\infty} \left( \tilde{P}_{n,i;k,0}(x)\ell_{S_{k,0}}(x) - \tilde{P}_{n,i;k,1}(x)\ell_{S_{k,1}}(x) \right) -
\]

\[
- \sum_{k=1}^{\infty} \left( \langle \lambda_{n,i} - \lambda_{k,0} \rangle S_{k,0}(x) - \langle \lambda_{n,i} - \lambda_{k,1} \rangle S_{k,1}(x) \right)
- \sum_{j=0}^{1} (-1)^j \sum_{m_{k,j} > 1} \sum_{s=0}^{m_{k,j} - 2} \tilde{P}_{n,i;k,s+1,j}(x) S_{k+s,j}(x),
\]
\[
\lambda_{n,i} \tilde{S}_n(x) + \tilde{S}_{n+1,i}(x) = \ell S_{n,i}(x) - \sum_{k=1}^{\infty} \left( \tilde{P}_{n+i;k,0} \ell S_{k,0} - \tilde{P}_{n+i;k,1} \ell S_{k,1} \right) - \\
- \sum_{k=1}^{\infty} \left( \left( \lambda_{n,i} - \lambda_{k,0} \right) \tilde{P}_{n+i;k,0} \ell S_{k,0} + \tilde{P}_{n+i-1;k,0} \ell S_{k,0} \right) - \\
- \left( \left( \lambda_{n,i} - \lambda_{k,1} \right) \tilde{P}_{n+i;k,1} \ell S_{k,1} + \tilde{P}_{n+i-1;k,1} \ell S_{k,1} \right) + \\
+ \frac{1}{\nu} \sum_{m=0}^{m_{n,i} - 1} \sum_{k=0}^{m_{n,i} - 1} \tilde{P}_{n+i;k+s+1,j} \ell S_{k+s,j}, \quad \nu = 1, m_{n,i} - 1,
\]

and consequently we arrive at

\[
\gamma_{n,i}(x) = \sum_{k=1}^{\infty} \left( \tilde{P}_{n+i;k,0} \gamma_{k,0} - \tilde{P}_{n+i;k,1} \gamma_{k,1} \right), \quad (n, i) \in \nu, \tag{80}
\]

where for \( l \in S_i, \nu = 1, m_{n,i} - 1 \)

\[
\gamma_{l,i}(x) := \ell S_{l,i}(x) - \lambda_{l,i} S_{l,i}(x), \quad \gamma_{l+i,v,i}(x) := \ell S_{l+i,v,i}(x) - \lambda_{l,i} S_{l+i,v,i}(x) - S_{l+i-1,v,i}(x).
\]

Using (80) we get

\[
\beta_{n,i}(x) = \sum_{k,j} \tilde{H}_{n,i;k,j} \beta_{k,j}(x), \quad (n, i), (k, j) \in \nu, \tag{81}
\]

where

\[
\beta_{n,1}(x) = n \gamma_{n,1}(x), \quad \beta_{n,0}(x) = n \chi_n(\gamma_{n,0}(x) - \gamma_{n,1}(x)).
\]

Since \(|\gamma_{n,i}(x)| \leq C n, \quad |\gamma_{n,0}(x) - \gamma_{n,1}(x)| \leq C n \chi_n, \) we have

\[
|\beta_{n,i}(x)| \leq C n^2. \tag{82}
\]

It follows from (50), (73), (81) and (82) that \(|\beta_{n,i}(x)| \leq C. \) Then, by virtue of Condition \( S \) in Theorem 6, \( \beta_{n,i}(x) = 0, \) and consequently \( \gamma_{n,i}(x) = 0. \) Thus, we obtain (70).

Furthermore, since

\[
\langle \tilde{S}(x), \tilde{B}_{n+i,v}(x) \rangle = \begin{cases} 
(\lambda - \lambda_{n,i}) A_{n+m_{n,i}+1}^{n+1}(x, \lambda), & \nu = m_{n,i} - 1, \\
(\lambda - \lambda_{n,i}) A_{n+1}^{n+1}(x, \lambda) - A_{n+1}^{n+1}(x, \lambda), & \nu = 0, m_{n,i} - 2,
\end{cases}
\]

formula (78) gives

\[
\lambda \tilde{S}(x) = \ell \tilde{S}(x, \lambda) - \lambda \sum_{k=1}^{\infty} \left( A_{k,0}(x, \lambda) S_{k,0}(x) - A_{k,1}(x, \lambda) S_{k,1}(x) \right).
\]

From this, by virtue of (46), it follows that \( \ell \tilde{S}(x, \lambda) = \lambda \tilde{S}(x, \lambda). \) Analogously using (79) we obtain \( \ell \tilde{F}(x, \lambda) = \lambda \tilde{F}(x, \lambda). \) Thus, (70), (71) are proved for the case when (73) is fulfilled.

Denote \( \Delta(\lambda) := S(T, \lambda). \) It follows from (46), (47) for \( x = T \) that

\[
\tilde{\Delta}(\lambda) = \Delta(\lambda) - \sum_{k=0}^{\infty} \left( A_{k,0}(T, \lambda) \Delta_{k,0} - A_{k,1}(T, \lambda) \Delta_{k,1} \right), \tag{83}
\]
0 = \Phi(T, \lambda) - \sum_{k=0}^{\infty} \left( \tilde{F}_{k,0}(T, \lambda) \Delta_{k,0} - \tilde{F}_{k,1}(T, \lambda) \Delta_{k,1} \right), \quad (84)

where \( \Delta_{n+\nu,1} = \Delta^{(\nu)}(\lambda_{n,1}) / \nu! \), \( n \in S_i \), \( \nu = 0, m_{n,1} - 1 \). Differentiating (83) with respect to \( \lambda \) an appropriate number of times and substituting \( \lambda = \lambda_{n,1} \) we get

\[
\Delta_{n,1} = \sum_{k=1}^{\infty} \left( \tilde{P}_{n,1;k,0}(T) \Delta_{k,0} - \tilde{P}_{n,1;k,1}(T) \Delta_{k,1} \right). \quad (85)
\]

Let us show that

\[
\tilde{P}_{n,1;k,1}(T) = -\delta_{n,k}. \quad (86)
\]

Indeed, for \( n, k \in \tilde{S}, \ \nu = 0, \tilde{m}_n - 1, \ s = 0, \tilde{m}_k - 1 \) we have

\[
\tilde{P}_{n+\nu,1;k+s,1}(T) = \sum_{p=0}^{\tilde{m}_n - 1 - s} \tilde{M}_{k+p+s} \tilde{D}_{\nu,p}(T, \tilde{\lambda}_n, \tilde{\lambda}_k). \quad (87)
\]

Moreover, according to (9) we have \( \langle \tilde{S}_\nu(x, \tilde{\lambda}_n), \tilde{S}(x, \tilde{\lambda}_k) \rangle |_{x=T} = 0 \), and hence

\[
(\tilde{\lambda}_n - \tilde{\lambda}_k) \tilde{D}_{\nu,p}(T, \tilde{\lambda}_n, \tilde{\lambda}_k) + \tilde{D}_{\nu-1,p}(T, \tilde{\lambda}_n, \tilde{\lambda}_k) - \tilde{D}_{\nu-1,0}(T, \tilde{\lambda}_n, \tilde{\lambda}_k) = 0, \quad (88)
\]

where \( \tilde{D}_{\alpha,\beta}(T, \tilde{\lambda}_n, \tilde{\lambda}_k) = 0 \) for negative \( \alpha \) or \( \beta \). For \( \tilde{\lambda}_n \neq \tilde{\lambda}_k \) solving the system (88) we obtain \( \tilde{D}_{\nu,p}(T, \tilde{\lambda}_n, \tilde{\lambda}_k) = 0 \), which together with (87) gives \( \tilde{P}_{n+\nu,1;k+s,1}(x) = 0 \) for \( n \neq k \). If \( \tilde{\lambda}_n = \tilde{\lambda}_k \), then (11), (88) give

\[
\tilde{D}_{\nu,p}(T, \tilde{\lambda}_n, \tilde{\lambda}_n) = \begin{cases} 
0, & 0 \leq \nu + p \leq \tilde{m}_n - 2, \\
\tilde{\alpha}_{n+\nu+p-\tilde{m}_n+1}, & \tilde{m}_n - 1 \leq \nu + p \leq 2\tilde{m}_n - 2. 
\end{cases} \quad (89)
\]

According to (87), (89) we have \( \tilde{P}_{n+\nu,1;n+s,1}(T) = 0, \ \nu < s \). Moreover, using (87), (89) and (28), we calculate

\[
\tilde{P}_{n+\nu,1;n+s,1}(T) = \sum_{p=0}^{\nu-s} \tilde{\alpha}_{n+\nu-s-p} \tilde{M}_{n+\nu+p-\tilde{m}_n+1} = -\delta_{\nu-s,0}
\]

and arrive at (86). Using (85), (86) we get

\[
\sum_{k=1}^{\infty} \tilde{P}_{n,1;k,0}(T) \Delta_{k,0} = 0.
\]

Then, by virtue of Condition \( S \), \( \Delta_{k,0} = 0, \ k \geq 1 \). Substituting this into (84) and using the relation \( \tilde{F}_{k,1}(T, \lambda) = 0, \ k \geq 1 \), we obtain \( \Phi(T, \lambda) = 0 \).

3) Let us now consider the general case when instead of (73) only (55) holds. Put

\[
\rho_{n,(l)} := \begin{cases} 
\rho_n, & n < l, \\
\tilde{\rho}_n, & n \geq l,
\end{cases} \quad M_{n,(l)} := \begin{cases} 
M_n, & n < l, \\
\tilde{M}_n, & n \geq l.
\end{cases}
\]

We agree that if the symbol \( \gamma \) denotes an object constructed with the help of the numbers \( \{\rho_n, M_n\}_{n \geq 1} \), then the symbol \( \gamma_{(l)} \) denotes the corresponding object constructed with the help of \( \{\rho_{n,(l)}, M_{n,(l)}\}_{n \geq 1} \). Then for all \( l \geq 1 \) we have

\[
\Omega_{1,(l)} = \left( \sum_{n=1}^{\infty} (n^2 \xi_{n,(l)})^2 \right)^{1/2} = \left( \sum_{n=1}^{l-1} (n^2 \xi_{n})^2 \right)^{1/2} < \infty.
\]
For each fixed \( l \geq 1 \) we solve the corresponding main equation:

\[
\hat{\phi}(l)(x) = (I - \tilde{H}(l))(x)\phi(l)(x),
\]

and construct the functions \( S(l)(x, \lambda) \) and the boundary value problem \( L(q(l)(x), T) \). Using Lemma 1.5.1 in [6] one can show that

\[
\lim_{l \to \infty} \|q(l) - q\|_{L^2} = 0, \quad \lim_{l \to \infty} \max_{0 \leq x \leq T} |S(l)(x, \lambda) - S(x, \lambda)| = 0.
\]

Denote by \( S_0(x, \lambda) \) the solution of equation (1) under the initial conditions \( S_0(0, \lambda) = 0, S'_0(0, \lambda) = 1 \). According to Lemma 1.5.3 in [6] we obtain

\[
\lim_{l \to \infty} \max_{0 \leq x \leq T} |S(l)(x, \lambda) - S_0(x, \lambda)| = 0.
\]

Hence \( S_0(x, \lambda) = S(x, \lambda) \), i.e. \( \ell S(x, \lambda) = \lambda S(x, \lambda) \). Similarly we get \( \ell \Phi(x, \lambda) = \lambda \Phi(x, \lambda) \).

Notice that we additionally proved that \( \Delta^{(\nu)}(\lambda_n) = 0, \nu = 0, m_n - 1, n \in \mathbb{S}, \) i.e. \( \{\lambda_n\}_{n \geq 1} \) is a spectrum of \( L \).

**Proof of Theorem 6.** According to (71), (72) the function \( \Phi(x, \lambda) \) is the Weyl function for the constructed boundary value problem \( L \). Choose \( n^* \) so that \( m_n = \tilde{m}_n = 1, n \geq n^* \), and put. Differentiating (47) with respect to \( x \) and then substituting \( x = 0 \), we obtain

\[
M(\lambda) = \tilde{M}(\lambda) + \sum_{k=n^*}^{\infty} \left( \frac{M_k}{\lambda - \lambda_k} - \frac{M_k}{\lambda - \lambda_k} \right) + \\
+ \sum_{k \in \mathbb{S}} \sum_{k < n^*}^{m_k - 1} \frac{M_{k+\nu}}{(\lambda - \lambda_k)^{\nu + 1}} - \sum_{k \in \mathbb{S}} \sum_{k < n^*}^{\tilde{m}_k - 1} \frac{\tilde{M}_{k+\nu}}{(\lambda - \lambda_k)^{\nu + 1}},
\]

(90)

where the series converges uniformly with respect to \( \lambda \) in bounded sets. From (61), (90) it follows that for each \( n \in \mathbb{S} \) the number \( \lambda_n \) is a pole of the function \( M(\lambda) \) of order \( m_n \). Thus, \( \{\lambda_n\}_{n \geq 1} \) is the spectrum and \( \{M_n\}_{n \geq 1} \) is the Weyl sequence of \( L \). Consequently \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) are the spectral data of \( L \).

**7. SPECIAL CASES AND STABILITY OF THE SOLUTION**

The requirement that the main equation is uniquely solvable (Condition \( S \) in Theorem 6) is essential and cannot be omitted (see Example 1.6.1 in [6]). Condition \( S \) is difficult to check in the general case. We point out three cases, for which the unique solvability of the main equation can be proved or checked.

1) The selfadjoint case. It is known that in the selfadjoint case, i.e. when the function \( q(x) \) is real-valued, the spectral data \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) are real numbers, and

\[
\lambda_n \neq \lambda_m (n \neq m), \quad \alpha_n > 0.
\]

(91)

Let real numbers \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) having the asymptotics (7), (19) with \( \{\kappa_n\} \in l_2 \) and satisfying (91) be given. Choose \( L \), construct \( \hat{\phi}(x) \), \( \hat{H}(x) \) and consider the equation (53). Similarly to Lemma 1.6.6 in [6] one can prove the following assertion.

**Lemma 8.** For each fixed \( x \in [0, T] \), the operator \( I - \hat{H}(x) \), acting from \( B \) to \( B \), has a bounded inverse operator. Thus, the main equation (53) has a unique solution \( \phi(x) \in B \).
By virtue of Theorem 6 and Lemma 8 the following theorem holds.

Theorem 7. For real numbers \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) to be the spectral data of a certain selfadjoint boundary value problem \( L(q(x), T) \) with \( q(x) \in L_2(0, T) \) it is necessary and sufficient to satisfy the asymptotics (7), (19) with \( \{\kappa_k\} \in l_2 \) and the condition (91).

2) Finite-dimensional perturbations of the spectral data. Let a model boundary value problem \( \tilde{L} \) with the spectral data \( \{\tilde{\lambda}_n, \tilde{\alpha}_n\}_{n \geq 1} \) be given. We change a finite subset of these numbers. In other words, we consider numbers \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) such that \( \lambda_n = \tilde{\lambda}_n, \alpha_n = \tilde{\alpha}_n, \) \( n > N, \) for certain \( N + 1 \in \mathbb{S} \) and arbitrary in the rest. Then for such spectral data the main equation becomes the linear algebraic system (54) and Condition S is equivalent to the condition that the determinant of this system does not equal to zero for each \( x \in [0, T] \). Such perturbations are very popular in applications. We note that for the selfadjoint case the determinant of the system (54) is always nonzero.

3) Local solvability of the main equation. For small perturbations of the spectral data Condition S is fulfilled automatically. Let us for simplicity consider the case of simple spectra, i.e. \( \mathbb{S} = \mathbb{N} \). The following theorem is valid.

Theorem 8. Let \( \hat{L} = L(\hat{q}(x), T) \) be given. There exists \( \delta > 0 \) (which depends on \( \hat{L} \)) such that if complex numbers \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) satisfy the condition \( \Omega < \delta \), then there exists a unique boundary value problem \( L(q(x), T) \) with \( q(x) \in L_2(0, T) \), for which the numbers \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) are the spectral data, and

\[
\|q - \hat{q}\|_{L_2(0,T)} < C\Omega,
\]

where \( C \) depends only on \( \hat{L} \).

Proof. Let \( C \) denote various constants which depend only on \( \hat{L} \). Since \( \Omega < \infty \), the asymptotical formulae (7), (19) are fulfilled. Choose \( \delta_0 \in (0, 1) \) such that if \( \Omega < \delta_0 \) then \( \alpha_n \neq 0, n \in \mathbb{S} \). According to (52) we have \( \|\hat{H}(x)\| \leq C\Omega \). Choose \( \delta \leq \delta_0 \) such that if \( \Omega < \delta \), then \( \|\hat{H}(x)\| \leq 1/2 \) for \( x \in [0, T] \). In this case there exists \( (I - \hat{H}(x))^{-1} \). Thus, all conditions of Theorem 6 are fulfilled, and hence there exists a unique \( q(x) \in L_2(0,T) \), such that the numbers \( \{\lambda_n, \alpha_n\}_{n \geq 1} \) are the spectral data of \( L(q(x), T) \). Moreover, (41) and (68) are valid. Using (57) one can get (92). \( \square \)

Theorem 8 gives the stability of Inverse Problem 1. Denote

\[
\Omega' := \left( \sum_{n=1}^{\infty} (n\xi'_n)^2 \right)^{1/2}
\]

where the numbers \( \xi'_n, \) \( n \geq 1 \), are determined by the formulae

\[
\xi'_{k+\nu} := \frac{|\lambda_k - \hat{\lambda}_k|}{k} + k^2 \sum_{p=0}^{m_k-1-n} |\alpha_{k+p} - \hat{\alpha}_{k+p}|
\]

for \( k \in \mathbb{S} \cap \tilde{\mathbb{S}}, \) \( m_k = \tilde{m}_k, \) \( \nu = 0, m_k - 1, \) and \( \xi'_n := 1 \) for other \( n \). According to (7), (19), (28) we have \( C_1\Omega \leq \Omega' \leq C_2\Omega, \) \( C_1, C_2 > 0, \) and hence (92) is equivalent to the estimate

\[
\|q - \hat{q}\|_{L_2(0,T)} < C\Omega'.
\]

Similarly to [6] one can obtain the stability of the solution in the uniform norm.

4) The case of integrable potentials. Using the method of spectral mappings one can solve the inverse problem also in \( L_1(0,T) \). The following theorem holds.
Theorem 9. For complex numbers $\{\lambda_n, \alpha_n\}_{n\geq 1}$ to be the spectral data for a certain $L(q(x), T)$ with $q(x) \in D \subset L_1(0, T)$, it is necessary and sufficient that

(i) (Asymptotics) there exists $\tilde{q}(x) \in D$ such that $\{n\xi_n\} \in l_2$;

(ii) $\alpha_n \neq 0$ for all $n \in \mathbb{S}$;

(iii) (Condition S) for each $x \in [0, T]$ the operator $I - \tilde{H}(x)$ has a bounded inverse one;

(iv) $\varepsilon(x) \in D$, where the function $\varepsilon(x)$ is defined by (57).

The boundary value problem $L(q(x), T)$ can be constructed by Algorithms 1, 2.

Similarly to [6] one can obtain necessary and sufficient conditions of the solvability for the inverse problem also for $q(x) \in W_N^2[0, T]$.

REFERENCES


Sergey Buterin
Department of Mathematics, Saratov State University
Astrakhanskaya str. 83, 410012 Saratov, Russia
E-mail: buterinsa@info.sgu.ru