Recursive Constructions for 3-Designs and Resolvable 3-Designs

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Dedicated to S. S. Shrikhande.

Abstract

Inspired by the doubling construction method for Steiner quadruple systems and also by a construction of Driessen for 3-designs, we present several recursive constructions for 3-designs and resolvable 3-designs. The construction methods assume the existence of resolvable 3-designs and certain appropriate other 3-designs. They prove to be very useful, as we can construct a large number of new infinite families of 3-designs. Among others we prove, for instance, that for any integer \( n \geq 3 \), there is a family \( \mathcal{F}_n \) of resolvable 3-designs having parameters \( 3 \) \(-\( v, k, \lambda \)\) design \( D \) here means that the blocks of \( D \) can be partitioned into parallel classes, each class consists of \( v/k \) pairwise disjoint blocks. For notation and definitions of \( t \)-designs we refer to [5]. Our aim is to present recursive methods for constructing 3-designs and resolvable 3-designs. Our constructions are inspired by the doubling construction for Steiner quadruple systems, which goes as far back as Witt (1938) [9] and a construction of Driessen [4] for 3-designs which can be considered as a generalization of the doubling construction. The paper is organized as follows. Construction I in section 2 is a general form of the doubling construction for 3-designs. The method turns out to be useful as many new families of 3-designs, which are presented in subsection 2.1, are constructed using this procedure. Constructions of resolvable 3-designs are shown in subsection 2.2, wherein applications of Construction I and further methods are explored, and many new families of resolvable 3-designs are displayed. Construction II in section 3 and Construction III in section 4 are methods which provide 3-designs whose number of points is not necessarily divisible by the block size.

In section 5 we show three special constructions for 3-designs with block sizes 5, 7 and 8. The paper is closed with an Appendix containing a list of parameters for newly constructed 3-designs.

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1 Introduction

All the designs considered in this paper are simple, i.e., no repeated blocks are allowed. A resolvable \( t-(v, k, \lambda) \) design \( D \) here means that the blocks of \( D \) can be partitioned into parallel classes, each class consists of \( v/k \) pairwise disjoint blocks. For notation and definitions of \( t \)-designs we refer to [5]. Our aim is to present recursive methods for constructing 3-designs and resolvable 3-designs. Our constructions are inspired by the doubling construction for Steiner quadruple systems, which goes as far back as Witt (1938) [9] and a construction of Driessen [4] for 3-designs which can be considered as a generalization of the doubling construction. The paper is organized as follows. Construction I in section 2 is a general form of the doubling construction for 3-designs. The method turns out to be useful as many new families of 3-designs, which are presented in subsection 2.1, are constructed using this procedure. Constructions of resolvable 3-designs are shown in subsection 2.2, wherein applications of Construction I and further methods are explored, and many new families of resolvable 3-designs are displayed. Construction II in section 3 and Construction III in section 4 are methods which provide 3-designs whose number of points is not necessarily divisible by the block size.

In section 5 we show three special constructions for 3-designs with block sizes 5, 7 and 8. The paper is closed with an Appendix containing a list of parameters for newly constructed 3-designs.
2 Construction I

The construction in this section is a most natural generalization of the doubling construction for Steiner quadruple systems.

Let $D = (X, B)$ be a resolvable $3 - (v, k, \lambda)$ (resp. $2 - (v, 2, 1)$) design, for $k \geq 3$ (resp. $k = 2$).

Let $\pi_1, \ldots, \pi_r$ denote the $r$ parallel classes of $D$. Define a distance between any two parallel classes $\pi_i$ and $\pi_j$ by $d(\pi_i, \pi_j) = \min \{ |i - j|, r - |i - j| \}$.

Let $\hat{D} = (\hat{X}, \hat{B})$ be a copy of $D$ such that $X \cap \hat{X} = \emptyset$. Let $D^* = (X, B^*)$ be a $3 - (v, 2k, \Lambda)$ design.

Define blocks on the point set $X \cup \hat{X}$ as follows:

I. blocks of a copy of $D^*$ defined on $X$;
II. blocks of a copy of $D^*$ defined on $\hat{X}$;
III. $B \cup \hat{B}$ for any pair $B \in \pi_i$ and $\hat{B} \in \pi_j$, with $\epsilon \leq d(\pi_i, \pi_j) \leq s$, $\epsilon = 0, 1$.

Case a: $k \geq 3$.

Any 3 points $a, b, c \in X$ (resp. $\hat{a}, \hat{b}, \hat{c} \in \hat{X}$) are contained in $\Lambda$ blocks of type I (resp. type II) and in $(2s + 1 - \epsilon) \lambda \frac{v}{k}$ of type III.

Any 3 points $a, b, c$, where $a, b \in X$ and $\hat{c} \in \hat{X}$, (resp. $\hat{a}, \hat{b}, c$) are contained in $(2s + 1 - \epsilon) \lambda \frac{v - 2}{k - 2}$ blocks of type III.

The defined blocks form a $3$-design if and only if $\Lambda + (2s + 1 - \epsilon) \lambda \frac{v}{k} = (2s + 1 - \epsilon) \lambda \frac{v - 2}{k - 2} \leq 2m - 1$, or equivalently $(2s + 1 - \epsilon) = \frac{\lambda v - 2 \Lambda}{\lambda v - 2 m}$. In this case we obtain a $3 - (2v, 2k, \frac{\lambda v - 2 \Lambda}{\lambda v - 2 m})$ design.

Case b: $k = 2$.

Here $D$ is the trivial $2 - (2m, 2, 1)$ design, and $D^*$ is a $3 - (2m, 4, \Lambda)$ design.

Any 3 points $a, b, c \in X$ (resp. $\hat{a}, \hat{b}, \hat{c} \in \hat{X}$) are contained in $\Lambda$ blocks of type I.

Any 3 points $a, b, c$, where $a, b \in X$ and $\hat{c} \in \hat{X}$, (resp. $\hat{a}, \hat{b}, c$) are contained in $(2s + 1 - \epsilon) \lambda \frac{v}{2}$ blocks of type III.

The condition for which the defined blocks form a $3$-design is $\Lambda = (2s + 1 - \epsilon) \leq 2m - 1$, and the constructed design has parameters $3 - (4m, 4, \Lambda)$.

It is clear from the construction that the resulting design is resolvable if $D^*$ is resolvable.

We summarize the construction in the following theorem.

Theorem 2.1 (i) If there exists a $3 - (2m, 4, \Lambda)$ design $D^*$ with $\Lambda \leq 2m - 1$, then there exists a $3 - (4m, 4, \Lambda)$ design $C$.

(ii) Suppose there exists a resolvable $3 - (v, k, \lambda)$ design $D$ and a $3 - (v, 2k, \Lambda)$ design $D^*$ such that $\frac{\Lambda v - 2 \Lambda}{2(\lambda v - 2 m)}$ is an integer $\leq r$, where $r$ is the number of parallel classes of $D$, then there exists a $3 - (2v, 2k, \Theta)$ design $C$ with $\Theta = \frac{\Lambda v - 2 \Lambda}{2(\lambda v - 2 m)}$.

Moreover, if $D^*$ is resolvable, then $C$ is resolvable for both cases (i) and (ii).

Remark 2.1 If $D^*$ is chosen to be a $3 - (2m, 4, 1)$ design in Theorem 2.1 (i), then we have the doubling construction for Steiner quadruple systems.

If $D^*$ is the trivial design in Theorem 2.1 and $\epsilon = 0$, then we have the construction of Driessen. It should be noted that the Driessen construction provides at most one 3-design
from a given resolvable \(3-(v, k, \lambda)\) design, whereas Construction I may yield a large number of \(3\)-designs from a given one. As an example, take the trivial \(3-(12,3,1)\) design for \(D\) and a \(3-(12,6, m2)\) design for \(D^*\), where \(m \in \{1, 2, \ldots, 42\}\). The numerical condition of Theorem 2.1 is satisfied if \(3|m\). Thus the resulting design \(C\) with parameters \(3-(24, 6, 10m/3)\) is obtained for \(m = 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42\). The last value \(m = 42\) corresponds to the design in Driessen construction.

### 2.1 Applications of Construction I

As a first example, take the \(3-(15,3,1)\) design for \(D\) and a \(3-(15, 6, m20)\) design for \(D^*\), where \(m \in \{1, 2, \ldots, 11\}\). The condition that \(\frac{\Lambda(k+2)}{2M(v-k)}\) is an integer implies that \(m\) is even. Hence the parameters of the resulting designs \(C\) are \(3-(30,6,65)\), \(3-(30,6,130)\), \(3-(30,6,195)\), \(3-(30,6,260)\) and \(3-(30,6,325)\). These designs are indicated as unknown in the Handbook of Combinatorial Designs [5], p.57.

Thus we have

**Theorem 2.2** There is a \(3-(30, 6, m5)\) design for \(m = 13, 26, 39, 52, 65\).

In the same vein as Theorem 2.2 we can prove that \(3-(32,8, m7)\) designs exist for \(m = 1, \ldots, 35\) by taking \(D\) as a resolvable \(3-(16,4,1)\) design and \(D^*\) as a \(3-(16,8, m3)\) design with \(m = 1, \ldots, 35\). Similarly, when \(D\) is a resolvable \(3-(20,4,1)\) design and \(D^*\) is a \(3-(20,8, m14)\) design, where \(m = 1, \ldots, 16\), the design \(C\) of parameters \(3-(40, 8, n63)\) can be constructed for all \(m = 2n\) with \(n = 1, \ldots, 8\).

Hence we have the following results.

**Theorem 2.3** (i) There exists a \(3-(32, 8, m7)\) design for \(m = 1, \ldots, 35\).

(ii) There exists a \(3-(40, 8, n63)\) design for \(n = 1, \ldots, 8\).

As another example, take the trivial \(3-(2^n + 1,3,1)\) design for \(D\), where \(n\) is odd. \(D\) is resolvable after a theorem of Baranyai [2]. Take \(D^*\) as a \(3-(2^n + 1, 6, 10(2^n - 2)/3)\) design with odd \(n \geq 5\). \(D^*\) is obtained from a \(4-(2^n + 1, 6, 10)\) design constructed by Bierbrauer [3]. It is easy to check that \(\frac{\Lambda(k+2)}{2M(v-k)} = (2s + 1 - \epsilon) = 5\) (i.e. \(\epsilon = 0\)). Theorem 2.1 then yields a \(3-(2^{n+1} + 2, 6, 5(2^n - 1))\) design. Thus we have the following result.

**Theorem 2.4** There exists a \(3-(2^{n+1} + 2, 6, 5(2^n - 1))\) design for all odd \(n \geq 5\).

We observe that the construction in Theorem 2.1 can produce infinite families of \(3\)-designs when using it recursively.

As examples we illustrate the construction of two families of \(3\)-designs with \(k = 8\).

1. Let \(D_i\) be a resolvable \(3-(2^i20, 4, 1)\) design for \(i \geq 0\). \(D_i\) is known to exist for all \(i\), see [5] I.4.32. Let \(D_0^*\) be a \(3-(20, 8, 28)\) design. Construction I with the pair \((D_0, D_0^*)\) yields a \(3-(40, 8, 63)\) design \(D_1^*\). Applying Construction I for the pair \((D_1, D_1^*)\) yields a \(3-(80, 8, 133)\) design \(D_2^*\). Repeat Construction I with the pair \((D_2, D_2^*)\) and so on will provide a family of \(3\)-designs having parameters \(3-(2^i20, 8, 7(2^{i-2}20 - 1))\) for all integers \(i \geq 0\). To see this, we need to verify the divisibility condition for \(\frac{\Lambda(0)(k+2)}{2\lambda(v-k)}\) and to compute \(\Lambda(i)\). Since \(v_i = 2^i20\),

\[
\Lambda(0) = 1 \text{ and } \Lambda(0) = \frac{\Lambda(0)(k+2)}{2\lambda(v-k)} = 7(v_i - 4)/4 = 7(2^{i-2}20 - 1) \text{ as desired.}
\]
2. In the same way, we will obtain a \(3 - (2^i28, 8, 7(2^i-228 - 1))\) design for all \(i \geq 0\) when starting with a resolvable \(3 - (28, 4, 1)\) design as \(D_0\) and a \(3 - (28, 8, 42)\) design as \(D_0^*\). Here we have \(\Lambda(i) = \frac{\Lambda^{(i-1)}_1(2(v_i-2))}{(v_i-1)}\) and \(\frac{\Lambda^{(0)}(k-2)}{\Lambda^{(0)}(k-4)} = 4\Lambda^{(0)}(1) = 4\Lambda^{(0)}(3) = \cdots = 4\Lambda^{(0)}(r) = 7.\) Thus we have proved the following result.

**Theorem 2.5** For all \(i \geq 0\) designs with the following parameters exist

1. \(3 - (2^i20, 8, 7(2^i-220 - 1))\),
2. \(3 - (2^i28, 8, 7(2^i-228 - 1))\).

### 2.2 Constructions of resolvable 3-designs

In this section we investigate constructions for resolvable 3-designs. As shown in Theorem 2.1 if both \(D\) and \(D^*\) are resolvable, then so is the resulting design \(C\). Whereas, if \(D^*\) is not resolvable, then, in general, \(C\) is not either. In the following, however, we prove that if \(v = 3k\) and \(D^*\), which is never resolvable in this case, is chosen in a particular way, then the resulting design \(C\) is resolvable. This result turns out to be very useful as it can be combined with Construction I to produce a great quantity of new families of resolvable 3-designs.

At first consider two simple but useful results related to resolvable \(t\)-designs with \(v = 2k\).

**Theorem 2.6** If there is a resolvable \(t - (2k, k, \lambda)\) design, then there is a resolvable \(t - (2k, k, \left(\frac{2k-r}{k-r}\right) - \lambda)\) design.

**Proof.** Let \(D\) be a resolvable \(t - (2k, k, \lambda)\) design, then the supplementary design \(\tilde{D}\) consisting of all \(k\)-subsets not being a block of \(D\) is a \(3 - (2k, k, \left(\frac{2k-r}{k-r}\right) - \lambda)\) design. \(\tilde{D}\) is resolvable, because if \(\tilde{C}\) is a block of \(\tilde{D}\), then the complement \(\tilde{C}^*\) is also a block of \(\tilde{D}\), since otherwise \(\tilde{C}^*\) and therefore \(\tilde{C}\), would be both blocks of \(D\), which is impossible. \(\Box\)

The next theorem about resolvable \(t\)-designs with \(v = 2k\) is derived from a construction of Alltop [1].

**Theorem 2.7** If there exists a \((2t + 1) - (2k, k, \lambda)\) design, then there exists a resolvable \((2t + 1) - (2k, k, \lambda)\) design.

**Proof.** Suppose that there is a \((2t + 1) - (2k, k, \lambda)\) design \(D = (X, B)\). If \(D\) is resolvable, then the theorem is proved. If not, let \(D_z = (X_z, B_1'), X_z = X - \{z\}\), be the derived design \(2t - (2k - 1, k - 1, \lambda)\) of \(D\) at a point \(z \in X\). Then \(D^* = (X_z \cup \{z\}, B_1^* \cup B_1^*)\) is a resolvable \((2t + 1) - (2k, k, \lambda)\) design, where \(B_1^* = B \cup \{z\}, B' \in B_1'\) and \(B_1^* = \{X - B, B \in B_1\}\). \(\Box\)

As an illustration of Theorem 2.6 and Theorem 2.7, we present several small parameters of resolvable \(3 - (2k, k, \lambda)\) designs for \(k \leq 10\) using known 3-designs given in [5].

**Theorem 2.8** There is a resolvable 3-design for the following parameters.

- \((i)\) \(3 - (8, 4, n), n = 1, \ldots, 5;\)
- \((ii)\) \(3 - (10, 5, n3), n = 1, \ldots, 7;\)
- \((iii)\) \(3 - (12, 6, n2), n = 1, \ldots, 42;\)
(iv) $3 - (14, 7, n_5), n = 1, \ldots, 66$;
(v) $3 - (16, 8, n_3), n = 1, \ldots, 429$;
(vi) $3 - (18, 9, n_7), n = 1, \ldots, 715$;
(vii) $3 - (20, 10, n_4), n = 1, \ldots, 4862$.

As first examples for resolvable 3-designs obtained from Construction I we have

**Theorem 2.9** There is a resolvable 3-design for the following parameters:

(i) $3 - (24, 6, n_{10}), n = 1, \ldots, 14$;
(ii) $3 - (32, 8, m_7), m = 1, \ldots, 35$.

**Proof.** (i) Take the trivial design $3 - (12,3,1)$ for $D$ and a resolvable $3 - (12, 6, m_2)$ design for $D^*$, where $m = 1, \ldots, 42$. It is easily checked that if $3|m$, then the resulting design $C$ has parameters $3 - (24, 6, \frac{m}{3}10)$.

(ii) In this case, $D$ is a resolvable Steiner quadruple system $3 - (16,4,1)$ and $D^*$ is a resolvable $3 - (16, 8, m_3), m = 1, \ldots, 35$. □

We now consider the case $v = 3k$ of Construction I.

Suppose there is a resolvable $3 - (3k, k, \lambda)$ design $D$, $k \geq 3$. Take the complementary design of $D$ for $D^*$. So, $D^*$ is a $3 - (3k, 2k, \Lambda)$ design with $\Lambda = \lambda(\frac{2k}{3}) / \binom{k}{3}$. Note that $D^*$ is never resolvable. It is now easy to verify that $\frac{\lambda(2k)}{2k(3k-1)} = 2k - 1$, hence Construction I yields a design $C$ with parameters $3 - (6k, 2k, \Theta)$, where $\Theta = \lambda(2k-1)(3k-2)/(k-2)$.

We show that $C$ is resolvable. Let $\tilde{D}^*$ be a copy of $D^*$ defined on $X$. First of all, note that $D$ and $D^*$ have the same number of blocks. Since $2s + 1 - \epsilon = 2k - 1$, we have $\epsilon = 0$.

**Type I**

Let $A_{i_i}, A_{i_2}, A_{i_3}$ (resp. $\tilde{A}_{i_1}, \tilde{A}_{i_2}, \tilde{A}_{i_3}$) be 3 blocks of the parallel class $\pi_i$ (resp. $\pi_i$). Let $B_{i_1}, B_{i_2}, B_{i_3}$ (resp. $\tilde{B}_{i_1}, \tilde{B}_{i_2}, \tilde{B}_{i_3}$) be the corresponding complementary blocks of $A_{i_j}$ in $D^*$ (resp. of $A_{i_j}$ in $\tilde{D}^*$).

Form 5 parallel classes of $C$ as follows.

$A_{i_1} \cup \tilde{A}_{i_1}$  $A_{i_2} \cup \tilde{A}_{i_2}$  $A_{i_3} \cup \tilde{A}_{i_3}$  $A_{i_1} \cup \tilde{A}_{i_2}$  $A_{i_1} \cup \tilde{A}_{i_3}$  $A_{i_2} \cup \tilde{A}_{i_3}$  $A_{i_2} \cup A_{i_3}$  $A_{i_1} \cup A_{i_2}$  $A_{i_1} \cup A_{i_3}$

It is clear that parallel classes of type I cover all the blocks of $D^*$ and $\tilde{D}^*$.

**Type II**

For each pair $(i, j)$, $i \neq j$ with $1 \leq d(\pi_i, \pi_j) \leq s$, the nine blocks of the form $A \cup \tilde{A}$, where $A \in \pi_i$ and $\tilde{A} \in \pi_j$, are partitioned into 3 parallel classes as follows.

$A_{i_1} \cup \tilde{A}_{j_1}$  $A_{i_2} \cup \tilde{A}_{j_2}$  $A_{i_3} \cup \tilde{A}_{j_3}$  $A_{i_1} \cup A_{j_2}$  $A_{i_2} \cup A_{j_3}$  $A_{i_3} \cup A_{j_1}$  $A_{i_1} \cup A_{j_3}$  $A_{i_2} \cup A_{j_1}$  $A_{i_3} \cup A_{j_2}$
This shows that $C$ is resolvable and has parameters $3 - (6k, 2k, \Theta)$, where $\Theta = \lambda(2k - 1)(3k - 2)/(k - 2)$.

Now, if we repeat the construction above with $C$, we obtain a further resolvable $3 - (12k, 4k, \Theta(4k - 1)(6k - 2)/(2k - 2))$ design. Continuing this procedure will provide a resolvable $3 - \left(2^i 3k, 2^i k, \lambda \prod_{j=0}^{i-1} \theta_j \right)$ design after $i$ steps of recursion, where $\theta_j = (2 \cdot 2^i k - 1)(3 \cdot 2^i k - 2)/(2^i k - 2)$.

Thus we have proved the following result.

**Theorem 2.10** If a resolvable $3 - (3k, k, \lambda)$ design with $k \geq 3$ exists, then a resolvable $3 - (6k, 2k, \lambda(2k - 1)(3k - 2)/(k - 2))$ design exists. In particular, there exists a resolvable $3 - \left(2^i 3k, 2^i k, \lambda \prod_{j=0}^{i-1} \theta_j \right)$ design for any $i \geq 1$, where $\Theta = \lambda \prod_{j=0}^{i-1} \theta_j$ and $\theta_j = (2 \cdot 2^i k - 1)(3 \cdot 2^i k - 2)/(2^i k - 2)$.

**Remark 2.2** If $k = 2$, then we start with the trivial $2 - (6,2,1)$ design $D$. The complementary design $D^*$ of $D$ is the trivial $3 - (6,4,3)$ design. The same argument in the proof of Theorem 2.10 shows that a resulting $3 - (12,4,3)$ design $C$ is resolvable. Therefore, the assumption $k \geq 3$ in Theorem 2.10 is not essential, it is made in order to avoid a zero division in the expression of $\theta_0$.

Theorem 2.1 seems to be a crucial and powerful tool for constructing resolvable 3-designs. First of all, the case $v = 2k$ provides the most known examples of resolvable 3-designs for infinitely many values of $k$, for instance Hadamard 3-designs. Furthermore, Theorem 2.6 finally asserts the abundance of resolvable $3 - (2k, k, \lambda)$ designs. Up to now very little was known about resolvable 3-designs with $v = 3k$. Theorem 2.10 is therefore interesting, because it can be used to show (for example) that non-trivial resolvable 3-designs with $v = 3k$ exist for infinitely many values of $k$. For any given value $k \geq 3$, applying Theorem 2.10 to the trivial resolvable $3 - (3k, k, \binom{3k-3}{k-3})$ yields an infinite family of resolvable 3-designs with parameters $3 - (2^i 3k, 2^i k, \binom{3k-3}{k-3} \prod_{j=0}^{i-1} \theta_j)$, where $\theta_j = (2 \cdot 2^i k - 1)(3 \cdot 2^i k - 2)/(2^i k - 2)$. It should be mentioned that these designs are non-trivial for all $i \geq 1$.

We record this result in the following theorem.

**Theorem 2.11** For any integer $k \geq 3$ there is a resolvable 3-design with parameters $3 - (2^i 3k, 2^i k, \binom{3k-3}{k-3} \prod_{j=0}^{i-1} \theta_j)$, where $\theta_j = (2 \cdot 2^i k - 1)(3 \cdot 2^i k - 2)/(2^i k - 2)$, for any $i \geq 1$.

Kramer and Magliveras [7] have shown the existence of 9 mutually disjoint copies of the $5 - (24, 8, 1)$ Witt design. The blocks of the $5 - (24, 8, 1)$ Witt design can be partitioned into 253 parallel classes each having three blocks, see for instance [8]. So we have a resolvable $3 - (24, 8, m21)$ design for $m = 1, \ldots, 9$. Starting Theorem 2.10 with each of these designs will provide a further family, which is presented in the following theorem.

**Theorem 2.12** For any $m = 1, \ldots, 9$ and $i \geq 1$, there is a resolvable 3-design with parameters $3 - (2^i 24, 2^i 8, m21 \prod_{j=0}^{i-1} \theta_j)$, where $\theta_j = (2^{i+4} - 1)(3 \cdot 2^{i+3} - 2)/(2^{i+3} - 2)$.

Before we discuss the combination of Theorem 2.1 and Theorem 2.10, we consider the construction of a family of resolvable 3-designs for $k = 8$ using Construction I.

Let $D_i$ be a resolvable $3 - (2^i 24, 4, 3)$ design, for all integer $i \geq 0$. For the existence of $D_i$, see [6]. Take a resolvable $3 - (24, 8, 105)$ design for $D_0$. Starting with the pair $(D_0, D_0^*)$ and applying Construction I repeatedly as shown above for the family in Theorem 2.5 we obtain a family of resolvable 3-designs with parameters $3 - (2^i 24, 8, 21(2^{i-2} 24 - 1))$. For instance,
$D^*_1$ (resp. $D^*_2$) has parameters $3 - (48, 8, 231)$ (resp. $3 - (96, 8, 483)$). To our knowledge this family is unknown.

We record the result in the following theorem.

**Theorem 2.13** There exists a resolvable $3 - (2^i 24, 8, 21(2^{i-2} 24 - 1))$ design for all integer $i \geq 0$.

We now show how to combine Theorems 2.1 and 2.10 by presenting a family of resolvable 3-designs for $k = 16$.

Let $D_1$ be a resolvable $3 - (2^i 24, 8, 21(2^{i-2} 24 - 1))$ design in Theorem 2.10. We start with $D_0$ as a 3-(24,8,105) design and $D_1^0$ as a 3-(24,16,1050) design, the complement of $D_0$. Then the constructed design $D^*_1$ has parameters $3 - (48, 16, 5775)$ and is resolvable. Applying Construction I to the pair $D_1$, $D^*_1$ yields a further resolvable design $D^*_2$. Continuing this way, the constructed design $D^*_3$ is resolvable and has parameters $3 - (2^i 24, 16, 7.15(2^{i-2} 24 - 1)(2^{i-3} 24 - 1))$. To verify this, we need to check the divisibility condition for $\frac{\lambda^{(i)}(k - 2)}{2^{2M(i)(v, k - 2)}}$, and to compute $\lambda^{(i)}$. Since $\lambda^{(i)} = \frac{\Lambda^{(i)}(v, k - 2)}{2^{2M(i)(v, k - 2)}}$ and $2\lambda^{(i)} = 21(v - 2)$, we have $\frac{\Lambda^{(i)}(k - 2)}{2^{2M(i)(v, k - 2)}} = \cdots = \frac{\Lambda^{(i)}(k - 2)}{2^{M(i)(v, k - 2)}} = (2k - 1) = 15$. Hence, $\lambda^{(i)} = 15.2\lambda^{(i)}(v, k - 2) = 7.15(2^{i-2} 24 - 1)(2^{i-3} 24 - 1))$ as desired.

We obtain the following result.

**Theorem 2.14** For any integer $j \geq 0$, there exists a resolvable $3 - (2^j 48, 16, 7.15(2^{j-2} 48 - 1))(2^{j-3} 48 - 1))$ design.

The construction of the family in Theorem 2.14 can be recursively carried out with respect to each given block size $2^n$, $n \geq 3$. In this way we obtain a double infinite family of resolvable 3-designs. In the following, we sketch this procedure.

For each $n \geq 3$, set $k_n = 2^n$ and $v_{n,j} = 2^j.2^n$.

Starting with the family of resolvable designs in Theorem 2.13: $3 - (v_{n,j}, k_n, \lambda^{n,j})$, where $\lambda^{n,j} = \frac{1}{2}(k - 2)(k - 1)(2^{j-2} 3.2^n - 1) = \frac{1}{2}(k - 2)(k - 1)(v_{n,j-2} - 1)$, we obtain a family of resolvable 3-designs in Theorem 2.14: $3 - (v_{n,j}, k_n, \lambda^{n,j})$ with $\lambda^{n,j} = \frac{1}{2}(k_n - 2)(k_n - 1)(v_{n,j} - 1)$, for all $j \geq 0$, by using Theorem 2.1 and 2.10, which will be called the combined procedure, or CP for short.

Now starting with the family: $3 - (v_{n,j}, k_n, \lambda^{n,j})$ and applying CP, we obtain a new family of resolvable designs: $3 - (v_{n,j}, k_n, \lambda^{n,j})$, with $\lambda^{n,j} = \frac{1}{2}(k_n - 2)(k_n - 1)(v_{n,j-1} - 1)(v_{n,j-2} - 1)$, for all $j \geq 0$.

When repeating the application of CP to the new family just constructed, we will obtain for each $n \geq 3$ a family of resolvable 3-designs having parameters $3 - (2^j 3.2^n, 2^n, \lambda^{n,j})$, for all $j \geq 0$, where $\lambda^{n,j} = (2^{n-1} - 1)(2^n - 1) \prod_{j=2}^{n-1} (2^{j-1} 3.2^n - 1)$.

The divisibility condition of Theorem 2.1 turns out to be $\lambda^{n,j} = k_n(2k_n - 2)/(2\lambda^{n,j}(v_{n,j-1} - k_n)) = 2k_n - 1$, by using the fact that $\lambda^{n,j} = \lambda^{n,j}(v_{n,j-1} - k_n) = 2k_n - 1$ and $\lambda^{n,j} = (2^{n-1} - 1)(2^n - 1) \prod_{j=2}^{n-1} (v_{n,j-1} - k_n)$.

We summarize this result in the following theorem.

**Theorem 2.15** Let $n \geq 3$ be an integer. Then there is a family $F_n$ of resolvable 3-designs having parameters $3 - (2^j 3.2^n, 2^n, (2^{n-1} - 1)(2^n - 1) \prod_{j=2}^{n-1} (2^{j-1} 3.2^n - 1))$, for all $j \geq 0$. 

7
3 Construction II

We have seen that Construction I provides a class of 3-designs for which the size of blocks divides the number of points. In this section, we want to extend Construction I so that we are able to construct designs for which the number of points is not necessarily divisible by the size of the blocks.

Let $D_1 = (X, B_1)$ be a resolvable $3 - (v, k_1, \lambda)$ design and let $D_2 = (X, B_2)$ be a resolvable $3 - (v, k_2, \zeta)$ design with $3 \leq k_1 < k_2$ such that $\lambda \frac{(v-1)(v-2)}{(k_1-1)(k_1-2)} = \zeta \frac{(v-1)(v-2)}{(k_2-1)(k_2-2)}$, i.e. $D_1$ and $D_2$ have the same number of parallel classes. Let $\pi_1, \ldots, \pi_r$ (resp. $\Pi_1, \ldots, \Pi_r$) denote the $r$ parallel classes of $D_1$ (resp. $D_2$).

Let $D_3 = (X, B_3)$ be a $3 - (v, k_1 + k_2, \Lambda)$ design and let $\tilde{D}_3$ be a copy of $D_i$, $i = 1, 2, 3$, constructed on the point set $\tilde{X}$ with $X \cap \tilde{X} = \emptyset$.

Define blocks on the point set $X \cup \tilde{X}$ as follows:

I. blocks of $D_3$ and $\tilde{D}_3$;

II. blocks of the form $A \cup \tilde{B}$, where $A \in \pi_i$ and $B \in \Pi_j$, such that $\epsilon \leq d(\pi_i, \pi_j) \leq s$, $\epsilon = 0, 1$;

III. blocks of the form $\tilde{A} \cup B$, where $\tilde{A} \in \tilde{\pi}_i$ and $B \in \Pi_j$, such that $\epsilon \leq d(\pi_i, \pi_j) \leq s$, $\epsilon = 0, 1$.

Any 3 points $a, b, c \in X$ (resp. $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$) are contained in, $\Lambda$ blocks of type I, $(2s + 1 - \epsilon) \lambda \frac{v}{k_2}$ blocks of type II and $(2s + 1 - \epsilon) \zeta \frac{v}{k_1}$ blocks of type III. Thus they appear in $\Lambda + (2s + 1 - \epsilon) \lambda \frac{v}{k_2} + (2s + 1 - \epsilon) \zeta \frac{v}{k_1}$ blocks.

We need to compute the number of blocks containing 3 points of type $a, b, \tilde{c}$, where $a, b \in X$ and $\tilde{c} \in \tilde{X}$; the case for three points $a, \tilde{b}, c$ is similar.

Now $a$ and $b$ are contained in $\lambda \frac{v-2}{k_2-2}$ blocks of $D_1$ and $\tilde{c}$ is in exactly one block of each parallel class of $D_2$. So $a, b, \tilde{c}$ are in $(2s + 1 - \epsilon) \lambda \frac{v-2}{k_2-2}$ blocks of type II. Similarly, $a, \tilde{b}, c$ are in $(2s + 1 - \epsilon) \zeta \frac{v-2}{k_1-2}$ blocks of type III. Thus $a, b, \tilde{c}$ are in $(2s + 1 - \epsilon) \lambda \frac{v-2}{k_2-2} + (2s + 1 - \epsilon) \zeta \frac{v-2}{k_1-2}$ blocks.

These defined blocks will form a 3-design if

$$\Lambda + (2s + 1 - \epsilon) \lambda \frac{v}{k_2} + (2s + 1 - \epsilon) \zeta \frac{v}{k_1} = (2s + 1 - \epsilon) \lambda \frac{v-2}{k_1-2} + (2s + 1 - \epsilon) \zeta \frac{v-2}{k_2-2}$$

or

$$\Lambda = \left[ \lambda \frac{v-2}{k_1-2} + \zeta \frac{v-2}{k_2-2} - (\lambda \frac{v}{k_2} + \zeta \frac{v}{k_1}) \right] (2s + 1 - \epsilon)$$

There are two cases:

Case A.

$$\lambda \frac{v-2}{k_1-2} + \zeta \frac{v-2}{k_2-2} - (\lambda \frac{v}{k_2} + \zeta \frac{v}{k_1}) = 0.$$ 

This implies $\Lambda = 0$ and the designs $D_3$ and $\tilde{D}_3$ are not needed in the construction. That means that the blocks of type II and III themselves form a design for $0 \leq s \leq \left\lfloor \frac{v}{2} \right\rfloor$. In this case, we can construct a $3 - (2v, k_1 + k_2, \Theta)$ design with $\Theta = \max \left( \lambda \frac{v-2}{k_1-2} + \zeta \frac{v-2}{k_2-2} \right)$ for any $m = 1, \ldots, r$.

Case B.

$$\lambda \frac{v-2}{k_1-2} + \zeta \frac{v-2}{k_2-2} - (\lambda \frac{v}{k_2} + \zeta \frac{v}{k_1}) > 0.$$
Here the defined blocks form a design if

\[ \Lambda / \left[ \frac{v - 2}{k_1} + \zeta \frac{v - 2}{k_2} - (\Lambda \frac{v}{k_2} + \zeta \frac{v}{k_1}) \right] = \Omega \]

is a positive integer \( \leq r \).

The parameters of the constructed design are \( 3 - (2v, k_1 + k_2, \Theta) \), where \( \Theta = \Omega \left( \Lambda \frac{v - 2}{k_1} + \zeta \frac{v - 2}{k_2} \right) \).

We summarize Construction II in the following theorem.

**Theorem 3.1** Suppose that there exists a resolvable \( 3 - (v, k_1, \lambda) \) design \( D_1 \) and a resolvable \( 3 - (v, k_2, \zeta) \) design \( D_2 \) with \( 3 \leq k_1 < k_2 \) such that \( \lambda \frac{(v - 1)(v - 2)}{(k_1 - 1)(k_1 - 2)} = \zeta \frac{(v - 1)(v - 2)}{(k_2 - 1)(k_2 - 2)} = r \).

(i) If \( \Lambda \frac{v - 2}{k_1} + \zeta \frac{v - 2}{k_2} - (\Lambda \frac{v}{k_2} + \zeta \frac{v}{k_1}) = 0 \), then there is a \( 3 - (2v, k_1 + k_2, \Theta) \) design with \( \Theta = m(\Lambda \frac{v - 2}{k_1} + \zeta \frac{v - 2}{k_2}) \) for any \( m = 1, \ldots, r \).

(ii) If \( \Lambda \frac{v - 2}{k_1} + \zeta \frac{v - 2}{k_2} - (\Lambda \frac{v}{k_2} + \zeta \frac{v}{k_1}) > 0 \) and if there is a \( 3 - (v, k_1 + k_2, \lambda) \) design \( D_3 \) such that

\[ \Lambda / \left[ \frac{v - 2}{k_1} + \zeta \frac{v - 2}{k_2} - (\Lambda \frac{v}{k_2} + \zeta \frac{v}{k_1}) \right] = \Omega \]

is a positive integer \( \leq r \), then there is a \( 3 - (2v, k_1 + k_2, \Theta) \) design with \( \Theta = \Omega \left( \Lambda \frac{v - 2}{k_1} + \zeta \frac{v - 2}{k_2} \right) \).

**Remark 3.1** If \( \Lambda \frac{v - 2}{k_1} + \zeta \frac{v - 2}{k_2} - (\Lambda \frac{v}{k_2} + \zeta \frac{v}{k_1}) < 0 \), then no design can be constructed.

As a first example of Construction II, take a resolvable 3-(12, 4, 3) design as \( D_1 \) and the resolvable 3-(12, 6, 10) design in Theorem 2.8 as \( D_2 \). Take the trivial 3-(12, 10, 36) design as \( D_3 \). Then Construction II yields a 3-(24, 10, 360) design. The latter is indicated as unknown in [5], p.55.

**Theorem 3.2** A \( 3 - (24, 10, 360) \) design exists.

As a second example consider a resolvable 3-(18, 6, 35) design \( D_1 \) and a resolvable 3-(18, 9, 98) design \( D_2 \). Note that \( D_1 \) is obtained from Theorem 2.10 by using the trivial 3-(9, 3, 1) design and \( D_2 \) is from Theorem 2.8. We have \( \Lambda \frac{v - 2}{k_1} + \zeta \frac{v - 2}{k_2} - (\Lambda \frac{v}{k_2} + \zeta \frac{v}{k_1}) = 0 \), and so there is \( 3 - (36, 15, m364) \) design for any \( m = 1, \ldots, 476 \).

**Theorem 3.3** There is a \( 3 - (36, 15, m364) \) design for any \( m = 1, \ldots, 476 \).

**4 Construction III**

In this section we present a further construction of 3-designs having block size not dividing the number of points.

Let \( T = (X, B_T) \) be a resolvable \( 3 - (v, \ell, \lambda) \) design. Let \( \pi_1, \ldots, \pi_r \) denote the \( r \) parallel classes of \( T \) where \( r = \lambda \frac{(v - 1)(v - 2)}{(v - 1)(v - 2)} \). As before, define a distance between any two parallel classes \( \pi_i \) and \( \pi_j \) of \( T \) by \( d(\pi_i, \pi_j) = \min \{|i - j|, r - |i - j|\} \).

Let \( \tilde{T} = (\tilde{X}, \tilde{B}_T) \) be a copy of \( T \) defined on \( \tilde{X} \) with \( \tilde{X} \cap X = \emptyset \). Let \( D = (X, B_D) \) be a \( 3 - (v, k, \lambda) \) design, such that \( \omega = k - \ell \geq 3 \). Let \( \tilde{D} \) be a copy of \( D \) defined on \( \tilde{X} \).
Further, let $W$ be a $3 - (\ell, w, \theta)$ design. We also assume that any two blocks of $T$ have less than $w$ points in common. This condition guarantees that the resulting design is simple; if this condition is removed then the constructed design may have repeated blocks.

Define blocks on the point set $X \cup \tilde{X}$ as follows:

I. blocks of $D$ and $\tilde{D};$

II. blocks of the form $B \cup \tilde{Z},$ where $B \in \pi_i$ and $\tilde{Z}$ is a block of the design $W$ defined on the points of a block in $\pi_j$ with $\epsilon \leq d(\pi_i, \pi_j) \leq s,$ $\epsilon = 0, 1;$

III. blocks of the form $\tilde{B} \cup Z,$ where $\tilde{B} \in \pi_i$ and $Z$ is a block of the design $W$ defined on the points of a block in $\pi_j$ with $\epsilon \leq d(\pi_i, \pi_j) \leq s,$ $\epsilon = 0, 1.$

Let $\{x, y, z\}$ be three points in $X.$

- $\{x, y, z\}$ are on $\Lambda$ blocks of type I.
- $\{x, y, z\}$ are on $\lambda$ blocks of $T$ distributed in $\lambda$ parallel classes $\pi_i.$ As there are $(2s + 1 - \epsilon)$ parallel classes $\pi_j$ satisfying $\epsilon \leq d(\pi_i, \pi_j) \leq s,$ and there are $v/\ell$ blocks in $\pi_j,$ there are $(2s + 1 - \epsilon)\lambda \theta_0 \frac{v}{\ell}$ choices for blocks of type II containing $\{x, y, z\},$ where $\theta_0$ is the number of blocks of $W.$
- There are $\lambda$ parallel classes $\pi_i$ having a block containing $\{x, y, z\}.$ In the copy of $W$ defined on the points of that block, $\{x, y, z\}$ are in $\theta$ blocks $Z.$ Thus there are $\lambda \theta$ choices for $Z.$ Further, there are $v/\ell$ blocks $\tilde{B}$ in $\pi_j$ with $\epsilon \leq d(\pi_i, \pi_j) \leq s,$ so there are $\lambda \theta (2s + 1 - \epsilon) \frac{v}{\ell}$ blocks of type III containing $\{x, y, z\}.$

Altogether, there are $\Lambda + \lambda (2s + 1 - \epsilon) \theta_0 \frac{v}{\ell} + \lambda (2s + 1 - \epsilon) \theta \frac{v}{\ell}$ blocks containing $\{x, y, z\}.$

Let $\{x, y, \tilde{z}\}$ be three points with $x, y \in X$ and $\tilde{z} \in \tilde{X}.$

- Two points $\{x, y\}$ are in $\lambda \frac{v - 2}{\ell - 2}$ blocks of $T$ distributed in $\lambda \frac{v - 2}{\ell - 2}$ parallel classes $\pi_i.$ For each of these $\pi_i,$ there are $(2s + 1 - \epsilon)$ choices for $\pi_j$ with $\epsilon \leq d(\pi_i, \pi_j) \leq s,$ and in $\pi_j$ there is a unique block containing $\tilde{z},$ so $\tilde{z}$ is in $\theta_1$ blocks $\tilde{Z}$ of $W$ defined on that block, where $\theta_1$ is the number of blocks containing a point in $W.$ Hence, there are $\lambda \theta_1 (2s + 1 - \epsilon) \frac{v - 2}{\ell - 2}$ blocks of type II containing $\{x, y, \tilde{z}\}.$

- Each of $\lambda \frac{v - 2}{\ell - 2}$ parallel classes $\pi_i,$ for which $\{x, y\}$ are on a block $B,$ gives $\theta_2$ blocks $Z$ containing $\{x, y\}$ in the copy of $W$ defined on $B,$ where $\theta_2$ is the number of blocks containing a pair of points in $W.$ Further, there is a unique block $\tilde{B}$ containing $\tilde{z}$ in $\pi_j$ with $\epsilon \leq d(\pi_i, \pi_j) \leq s,$ so there are $(2s + 1 - \epsilon)\lambda \frac{v - 2}{\ell - 2} \theta_2$ blocks of type III containing $\{x, y, \tilde{z}\}.$

Therefore, $\{x, y, \tilde{z}\}$ are in $\lambda \theta_1 (2s + 1 - \epsilon) \frac{v - 2}{\ell - 2} + (2s + 1 - \epsilon)\lambda \frac{v - 2}{\ell - 2} \theta_2$ blocks.

The blocks so constructed will form a $3$-design if

$$
\Lambda + \lambda (2s + 1 - \epsilon) \theta_0 \frac{v}{\ell} + \lambda (2s + 1 - \epsilon) \theta \frac{v}{\ell} = \lambda \theta_1 (2s + 1 - \epsilon) \frac{v - 2}{\ell - 2} + (2s + 1 - \epsilon)\lambda \frac{v - 2}{\ell - 2} \theta_2
$$

(2)
or equivalently,

\[ \Lambda/\lambda \theta \left[ \frac{v-2}{\ell-2} \left( \frac{\ell-1}{2} \right) + \frac{\ell - 2}{w-2} - \frac{v}{\ell} \left( \frac{\ell}{2} \right) + 1 \right] = (2s + 1 - \epsilon) \]

is an integer \( \leq r \). And the resulting design has parameters \( 3 - (2v, k, \Theta) \), where

\[ \Theta = (2s + 1 - \epsilon) \lambda \theta \frac{v-2}{\ell-2} \left( \frac{\ell-1}{2} \right) + \frac{\ell - 2}{w-2}. \]

We summarize Construction III in the following theorem.

**Theorem 4.1** Suppose that there exists a resolvable \( 3 - (v, \ell, \lambda) \) design \( T \) and a \( 3 - (v, k, \Lambda) \) design \( D \) with \( w = k - \ell \geq 3, k \leq 2\ell \), and \( |\Lambda \cap B| \leq w - 1 \) for any two distinct blocks \( \Lambda \) and \( B \) of \( T \). Suppose that there is a \( 3 - (\ell, w, \theta) \) design \( W \) such that

\[ \Lambda/\lambda \theta \left[ \frac{v-2}{\ell-2} \left( \frac{\ell-1}{2} \right) + \frac{\ell - 2}{w-2} - \frac{v}{\ell} \left( \frac{\ell}{2} \right) + 1 \right] = \Omega \]

is an integer \( \leq r \), where \( r \) is the number of parallel classes of \( T \). Then there exists a \( 3 - (2v, k, \Theta) \) design \( C \), where \( \Theta = \Omega \lambda \theta \frac{v-2}{\ell-2} \left( \frac{\ell-1}{2} \right) + \frac{\ell - 2}{w-2}. \)

As an application of Theorem 4.1 we have the following Corollary.

**Corollary 4.2** If there exists a \( 3 - (4n, 7, \Lambda) \) design for \( n \equiv 4, 8 \pmod{12} \) such that \( 5(n - 1)|\Lambda \) and \( \Lambda \leq 5(n - 1)\left(\frac{\ell-1}{2}\right)/3 \), then there exists a \( 3 - (8n, 7, \Lambda \frac{w-1}{n-1}) \) design.

**Proof.** Take a resolvable \( 3 - (4n, 4, 1) \) design as \( T \) and the trivial \( 3 - (4, 3, 1) \) design as \( W \) for Theorem 4.1. \( \square \)

An example derived from Theorem 4.1 is as follows. Let \( T \) be the resolvable \( 3-(24,8,21) \) design, which is the Witt system \( 5-(24,8,1) \), and let \( D \) be a \( 3-(24,15, m5.7.13) \) design, which is the complementary design of a \( 3-(24,9, m84) \) design with \( m \in \{1, \ldots, 101\} \). Let take \( W \) to be the trivial \( 3-(8,7,5) \) design. It follows that \( \Omega = \frac{2}{5}m \) is an integer if \( m = 2n \). In this case Theorem 4.1 yields a \( 3-(48,15, n5.7.11.13) \) design. It is known that a \( 3-(24,9, m84) \) exists for all even values of \( m \), see [5], p.55, so we have the following theorem.

**Theorem 4.3** There is a \( 3-(48,15, n5.7.11.13) \) design for \( n = 1, \ldots, 50 \).

5 Special Constructions for \( k = 5, 7, 8 \)

In this section we present three special constructions for \( 3 \)-designs with block sizes \( 5,7,8 \).

5.1 A construction for \( k = 5 \)

Let \( D = (X, B) \) be a \( 3-(2n, 5, \lambda) \) design. And let \( \tilde{D} = (\tilde{X}, \tilde{B}) \) be a copy of \( D \) with \( X \cap \tilde{X} = \emptyset \). Let \( T \) be the resolvable \( 2-(2n, 2, 1) \) design defined on \( X \). Let \( T_1, \ldots, T_{2n-1} \) denote the \( 2n-1 \) parallel classes of \( T \). Define blocks for a \( 3-(4n, 5, \Lambda) \) design on the point set \( X \cup \tilde{X} \) as follows:
I. blocks of $D$ (resp. blocks of $\tilde{D}$);

II. blocks of the form $\{a, b, c, d, e\}$ (resp. $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}\}$), where \( \{a, b\} \in T_h \), \( \{b, c\} \in T_i \), \( \{c, a\} \in T_j \), \( \{d, \tilde{e}\} \in \tilde{T}_e \) and \( \ell \in \{b, i, j\} \).

Any three points $a, b, c \in X$ (resp. $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$) are contained in $\lambda$ blocks of type I and $3n$ blocks of type II.

Any three points $a, b, c$ with $a, b \in X$ and $\tilde{d} \in \tilde{X}$, (resp. $\tilde{a}, \tilde{b}, \tilde{d}$) are contained in

- $3(2n - 2)$ blocks of type $\{a, b, c, d, e\}$: there are $(2n - 2)$ choices for $c$ and for each such $c$ there are 3 possibilities for $\tilde{e}$ such that $\{\tilde{d}, \tilde{e}\} \in \{\tilde{T}_h, \tilde{T}_i, \tilde{T}_j\}$;

- $(3n - 3)$ blocks of type $\{\tilde{e}, \tilde{d}, \tilde{c}, a, b\}$: if $\{a, b\} \in T_h$, then there are $(2n - 1)$ choices for $\tilde{c}$, exactly one of them gives $\{\tilde{c}, \tilde{d}\} \in \tilde{T}_h$ and hence there are $(2n - 2)$ possible choices for $\tilde{e}$; from the remaining $(2n - 2)$ possible choices for $\tilde{c}$ we have $\{\tilde{c}, \tilde{d}\} \in \tilde{T}_i \neq \tilde{T}_h$ and $\tilde{e}$ has to be chosen such that $\{\tilde{c}, \tilde{e}\} \in \tilde{T}_h$, so there are $(n - 1)$ choices for the pair $\{\tilde{c}, \tilde{e}\}$ as a block in $\tilde{T}_h$.

In summary, there are $3(2n - 2) + (3n - 3) = 9(n - 1)$ blocks containing $a, b, \tilde{d}$. The blocks so defined will form a $3$-design if and only if $\lambda + 3n = 9(n - 1)$, or equivalently $\lambda = 6n - 9$. The design constructed will have parameters $3 - (4n, 5, 9(n - 1))$. Hence, we have the following theorem.

**Theorem 5.1** If there is a $3 - (2n, 5, 6n - 9)$ design, then there is a $3 - (4n, 5, 9(n - 1))$ design.

**Examples 5.1** As an application, Theorem 5.1 shows the existence of a $3-(36,5,72)$ and a $3-(44,5,90)$ design since a $3-(18,5,45)$ and a $3-(22,5,57)$ design exist.

**Remark 5.1** In the Driessen construction [4], p.87, $D$ is the trivial $3 - (2n, 5, \binom{2n-3}{2})$ design. In this case, the only value $n$ for which a $3$-design can be constructed is $n = 5$, and the design obtained has parameters $3 - (20, 5, 36)$.

### 5.2 A construction for $k = 7$

Let $D = (X, B)$ be a $3 - (3n, 7, \lambda)$ design. And let $\tilde{D} = (\tilde{X}, \tilde{B})$ be a copy of $D$ with $X \cap \tilde{X} = \emptyset$.

Let $T$ be the resolvable $3 - (3n, 3, 1)$ design defined on $X$. Denote by $T_1, \ldots, T_l$ the parallel classes of $T$, where $r = \binom{3n-1}{2}$. Define blocks for a $3 - (6n, 7, \lambda)$ design on the point set $X \cup \tilde{X}$ as follows:

I. blocks of $D$ (resp. blocks of $\tilde{D}$);

II. sets of the form $\{a, b, c, d, e, f, \tilde{g}\}$ (resp. $\{\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, e, f, \tilde{g}\}$), where $\{a, b, c\} \in T_i$, $\{b, c, d\} \in T_{i_2}$, $\{c, d, a\} \in T_{i_3}$, $\{d, a, b\} \in T_{i_4}$, and $\{e, f, \tilde{g}\} \in T_j$ and $\tilde{j} \in \{i_1, i_2, i_3, i_4\}$.

Any three points $a, b, c \in X$ (resp. $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{X}$) are contained in

- $\lambda$ blocks of type I;

- $4n(3n - 3)$ blocks of form $\{a, b, c, d, e, f, \tilde{g}\}$: there are $(3n - 3)$ possible choices for $d$ and each such a choice determines 4 parallel classes $T_{i_1}$, $T_{i_2}$, $T_{i_3}$, and $T_{i_4}$, the points $e, f, \tilde{g}$ have to be chosen such that they form a block of $\tilde{T}_{j_1}$, $j = 1, 2, 3, 4$, so there are $4n(3n - 3)$ blocks containing $\{a, b, c\}$;
• \(n(3n - 3)\) blocks of form \(\{\bar{g}, \bar{f}, \bar{e}, \bar{d}, a, b, c\}\); if \(\{a, b, c\} \in T_j\), then some 3 points of \(\{\bar{g}, \bar{f}, \bar{e}, \bar{d}\}\) must be a block in \(T_j\), so there are \(n\) possible choices for those three points, the fourth point can be chosen in \((3n - 3)\) ways; thus there are \(n(3n - 3)\) blocks containing \(\{a, b, c\}\).

Hence there are \(\lambda + 4n(3n - 3) + n(3n - 3) = \lambda + 5n(3n - 3)\) blocks of type I and II containing \(a, b, c\).

Any three points \(a, b, \bar{e}\) with \(a, b \in X\) and \(\bar{e} \in \bar{X}\) are contained in

• \(6(3n - 2)(n - 1)\) blocks of form \(\{a, b, c, d, \bar{e}, \bar{f}, \bar{g}\}\); there are \((\binom{3n - 2}{2})\) possible choices for a pair \(\{c, d\}\), each choice determines 4 parallel classes, and two points \(\bar{f}, \bar{g}\) have to be chosen so that \(\{\bar{e}, \bar{f}, \bar{g}\}\) is a block in one of these 4 parallel classes, thus there are 4 choices for \(\{\bar{e}, \bar{f}, \bar{g}\}\); altogether we have \(4 \binom{I}{2} = 6(3n - 2)(n - 1)\) blocks of form \(\{a, b, c, d, \bar{e}, \bar{f}, \bar{g}\}\) containing \(a, b, \bar{e}\);

• \(4(3n - 2)(n - 1)\) blocks of form \(\{\bar{g}, \bar{f}, \bar{e}, \bar{d}, a, b, c\}\); for each of \((3n - 2)\) choices for \(c \neq a, b\) denote by \(T_j\) the parallel class containing \(\{a, b, c\}\) as a block; there is exactly one block of \(T_j\) containing \(\bar{e}\) and also two other points, say \(\bar{d}, \bar{f}\); the last point \(\bar{g}\) can be chosen in \((3n - 3)\) different ways, this gives \((3n - 2)(3n - 3)\) blocks; on the other hand, for any of \((3n - 2)\) choices for \(c\), there are \((n - 1)\) blocks of the form \(\{\bar{d}, \bar{f}, \bar{g}\}\) in \(T_j\), so this gives \((n - 1)(3n - 2)\) blocks; altogether there are \((3n - 2)(3n - 3) + (n - 1)(3n - 2) = 4(n - 1)(3n - 2)\) blocks of form \(\{\bar{g}, \bar{f}, \bar{e}, \bar{d}, a, b, c\}\) containing \(a, b, \bar{e}\).

The blocks constructed on \(X \cup \bar{X}\) will form a design if any 3 points of the form \(a, b, c\) and \(a, b, \bar{e}\) are contained in the same number of blocks, i.e. if the condition \(\lambda = 5(n - 1)(3n - 4)\). So, any three points of the constructed design are contained in \(\Lambda = 6(n - 1)(3n - 2) + 4(n - 1)(3n - 2) = 10(n - 1)(3n - 2)\) blocks. Therefore we have the following theorem.

**Theorem 5.2** If there is a 3-(3n, 7, 5(n-1)(3n-4)) design, then there is a 3-(6n, 7, 10(n−1)(3n−2)) design for all \(n \geq 0\).

As examples we see that if a 3-(21,7,510) (resp. 3-(30,7,1170)) design exists then there exists a 3-(42,7,1140) (resp. 3-(60,7,2520)) design.

5.3 A construction for \(k = 8\)

In the same vein as the construction for \(k = 7\), we may also construct designs for \(k = 8\) when using the trivial 3-(3n, 3, 1) design.

Let \(D = (X, B)\) be a 3-(3n, 8, \(\lambda\)) design. Let \(D = (X, \bar{B})\) be a copy of \(D\) with \(X \cap \bar{X} = \emptyset\). Again, let \(T\) be the resolvable 3-(3n, 3, 1) design defined on \(X\). Denote by \(T_1, \ldots, T_r\) the parallel classes of \(T\), where \(r = \binom{3n-1}{2}\). Define blocks for a 3-(6n, 8, \(\Lambda\)) design on the point set \(X \cup \bar{X}\) as follows:

I. blocks of \(D\) (resp. blocks of \(\bar{D}\));

II. blocks of the form \(\{a, b, c, d, e, \bar{f}, \bar{g}, \bar{h}\}\) (resp. \(\{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, f, g, h\}\)), having the property that if \(\{f, g, h\} \in T_i\) then there are three points \(\{x, y, z\} \subseteq \{a, b, c, d, e\}\) with \(\{x, y, z\} \in T_i\).

Any three points \(a, b, c \in X\) (resp. \(\bar{a}, \bar{b}, \bar{c} \in \bar{X}\)) are contained in
• $\Lambda$ blocks of type I;

• $10n\binom{3n-3}{2}$ blocks of form \{a, b, c, d, e, f, g, h\}: there are \binom{3n-3}{2} possible choices for a pair \{d, e\} and each choice determines 10 parallel classes $T_{ij}$, $j = 1, \ldots, 10$, each of these classes contains exactly one 3-subset of \{a, b, c, d, e\}; points $f, g, h$ have to be chosen such that they form a block of $T_{ij}$, this yields $10n\binom{3n-3}{2}$ blocks containing \{a, b, c\};

• $n\binom{3n-3}{2}$ blocks of form \{h, g, f, e, d, a, b, c\}: if \{a, b, c\} $\in T_i$, then some 3 points of \{h, g, f, e, d\} must be as a block in $T_i$, so there are $n$ possible choices for those 3 points, the other two points of \{h, g, f, e, d\} can be chosen \binom{3n-3}{2} ways; this yields $n\binom{3n-3}{2}$ blocks containing \{a, b, c\}.

Hence there are $\lambda + 10n\binom{3n-3}{2} + n\binom{3n-3}{2} = \lambda + 11n\binom{3n-3}{2}$ blocks of type I and II containing $a, b, c$.

Any three points $a, b, f$ with $a, b \in X$ and $f \in \bar{X}$ are contained in

• $10\binom{3n-2}{3}$ blocks of form \{a, b, c, d, e, f, g, h\}: there are $\binom{3n-2}{3}$ possible choices for a triple \{c, d, e\}; five points \{a, b, c, d, e\} determine 10 parallel classes, $T_{ij}$, $j = 1, \ldots, 10$, each of these classes contains exactly one 3-subset of \{a, b, c, d, e\}; and points $g, h$ have to be chosen so that \{f, g, h\} is a block of $T_{ij}$, so there are 10 choices for \{f, g, h\}, this gives $10\binom{3n-2}{3}$ blocks containing \{a, b, f\};

• $5\binom{3n-2}{3}$ blocks of form \{h, g, f, e, d, a, b, c\}: for each of $(3n-2)$ choices for $c \neq a, b$ let $T_i$ be the parallel class containing \{a, b, c\} as a block; there is exactly one block of $T_i$ containing $f$ and also two other points, say $d, e$; the other two points $g$ and $h$ can be chosen in \binom{3n-2}{2} different ways, this yields $(3n-2)(3n-3)/2 = 3\binom{3n-2}{3}$ blocks containing $a, b, f$; on the other hand, for any of $(3n-2)$ choices for $c$, there are $(n-1)$ blocks of form \{x, y, z\} in $T_i$, where \{x, y, z\} $\subseteq \{h, g, e, d\}$, and there are $(3n-4)$ possible choices for another point of \{h, g, e, d\}, this yields $(3n-2)(n-1)(3n-4)/2 = 2\binom{3n-2}{3}$ blocks containing $a, b, f$; altogether there are $3\binom{3n-2}{3} + 2\binom{3n-2}{3} = 5\binom{3n-2}{3}$ blocks of form \{h, g, f, e, d, a, b, c\} containing $a, b, f$.

The blocks constructed on $X \cup \bar{X}$ will form a design if any 3 points of the form $a, b, c$ and $a, b, f$ are contained in the same number of blocks, i.e. if the condition $\lambda + 11n\binom{3n-3}{2} = 10\binom{3n-2}{3} + 5\binom{3n-2}{3}$ is satisfied. Hence $\lambda = (2n-5)(3n-3)(3n-4)$.

So, any three points of the design constructed are contained in $\Lambda = 10\binom{3n-2}{3} + 5\binom{3n-2}{3} = 15\binom{3n-2}{3}$ blocks. Therefore we have the following theorem.

**Theorem 5.3** If there is a 3 - (3n, 8, (2n - 5)(3n - 3)(3n - 4)) design, then there is a 3 - (6n, 8, 15\binom{3n-2}{3}) design for all $n \geq 0$.

**Examples 5.2** There is a 3-(36, 8, 8400) (resp. 3-(48, 8, 23100)) design since there is a 3-(18, 8, 1470) (resp. 3-(24, 8, 4620)) design.

**6 Appendix**

The following table contains a list of parameters for 3-designs constructed from the recursive methods of the paper.
<table>
<thead>
<tr>
<th>Parameters</th>
<th>Comments</th>
<th>Theorems</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $3 - (30, 6, m5)$, $m = 13, 26, 39, 52, 65$</td>
<td></td>
<td>Thm. 2.2</td>
</tr>
<tr>
<td>2. $3 - (40, 8, m63)$, $m = 1, \ldots, 8$</td>
<td></td>
<td>Thm. 2.3</td>
</tr>
<tr>
<td>3. $3 - (2^{n+1} + 2, 6, 5(2^n - 1))$, odd $n \geq 5$</td>
<td></td>
<td>Thm. 2.4</td>
</tr>
<tr>
<td>4. $3 - (2^i20, 8, 7(2^j - 20 - 1)), i \geq 0$</td>
<td></td>
<td>Thm. 2.5</td>
</tr>
<tr>
<td>5. $3 - (2^i28, 8, 7(2^i - 28 - 1)), i \geq 0$</td>
<td></td>
<td>Thm. 2.5</td>
</tr>
<tr>
<td>6. $3 - (24, 6, m10), m = 1, \ldots, 14$</td>
<td>resolvable</td>
<td>Thm. 2.9</td>
</tr>
<tr>
<td>7. $3 - (32, 8, m7), m = 1, \ldots, 35$</td>
<td>resolvable</td>
<td>Thm. 2.9</td>
</tr>
<tr>
<td>8. $3 - (2^j3k, 2^i, \frac{(3k-3)}{2} \prod_{j=0}^{i-1} \theta_j)$, $\theta_j = (2.2^{j}k - 1)/(3^j - 2), i \geq 1$</td>
<td>resolvable</td>
<td>Thm. 2.11</td>
</tr>
<tr>
<td>9. $3 - (2^j24, 2^8, m21 \prod_{j=0}^{i-1} \theta_j), m = 1, \ldots, 9, i \geq 1$, $\theta_j = (2^{j+3} - 1)/(2^{j+3} - 2)$</td>
<td>resolvable</td>
<td>Thm. 2.12</td>
</tr>
<tr>
<td>10. $3 - (2^j24, 8, 21(2^j - 22 - 1)), i \geq 0$</td>
<td>resolvable</td>
<td>Thm. 2.13</td>
</tr>
<tr>
<td>11. $3 - (2^j48, 16, 7.15(2^{j-2}48 - 1)(2^{j-3}48 - 1)), j \geq 0$</td>
<td>resolvable</td>
<td>Thm. 2.14</td>
</tr>
<tr>
<td>12. $3 - (2^j3.2^n, 2^n, (2^n - 1)(2^n - 1) \prod_{i=2}^{i-1}(2^{j-i}3.2^n - 1))$, $j \geq 0, n \geq 3$</td>
<td>resolvable</td>
<td>Thm. 2.15</td>
</tr>
<tr>
<td>13. $3 - (36, 15, m364)$</td>
<td></td>
<td>Thm. 3.2</td>
</tr>
<tr>
<td>14. $3 - (36, 15, m364)$, $m = 1, \ldots, 476$</td>
<td></td>
<td>Thm. 3.3</td>
</tr>
<tr>
<td>15. $3 - (48, 15, m57.11.13)$, $m = 1, \ldots, 50$</td>
<td></td>
<td>Thm. 4.3</td>
</tr>
</tbody>
</table>

**Remark 6.1** Families 10 and 11 in the table are special cases of family 12 with $n = 3$ and 4.

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References


