The supersingular locus in Siegel modular varieties, and Deligne-Lusztig varieties

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(joint work with Chia-Fu Yu)

Let p be a prime number, and $g \geq 1$ an integer. We consider the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties, and the following variant, the Siegel modular variety $\mathcal{A}_{g,I}$ with Iwahori level structure at p, which is much less well understood. By definition, $\mathcal{A}_I$ is the space of isomorphism classes of tuples

$$(A_0 \to A_1 \to \cdots \to A_g, \lambda_0, \lambda_g, \eta),$$

where the A_i are abelian varieties of dimension g, the maps $A_i \to A_{i+1}$ are isogenies of degree p, λ_0 and λ_g are principal polarizations of A_0 and A_g, respectively, such that the pull-back of λ_g is $p\lambda_0$, and η is a level structure away from p. We consider these spaces over an algebraic closure k of $\mathbb{F}_p$. Both have dimension $g(g+1)/2$.

Inside $\mathcal{A}_g$, we have the *supersingular locus*, i. e. the closed subset of those abelian varieties which are isogeneous to a product of supersingular elliptic curves. There are a number of results describing the geometry of the supersingular locus. For instance, it was proved by Li and Oort [3] that the dimension of the supersingular locus is $[g^2/4]$. It is known to be connected if $g \geq 2$, and there is a formula for the number of irreducible components, in terms of a certain class number, also proved in loc. cit.

On the other hand, in the Iwahori case, currently very little is known about the supersingular locus. Even its dimension is known only for $g \leq 3$ (but see below). Note that the situation here is definitely more complicated than in the case of $\mathcal{A}_g$; as an example, in the case $g = 2$, the supersingular locus coincides with the p-rank 0 locus, but it is not contained in the closure of the p-rank 1 locus. In addition, it is not equi-dimensional (see [6, Prop. 6.3]).

The supersingular locus (and especially its cohomology) are interesting objects from the point of view of automorphic representations and the Langlands program.

On the other hand we have the Kottwitz-Rapoport stratification (KR stratification)

$$\mathcal{A}_I = \coprod_{x \in \mathrm{Adm}} \mathcal{A}_{I,x}$$

by locally closed subsets, which should be thought of as a stratification by singularities. It corresponds to the stratification by Schubert cells of the associated local model. In terms of abelian varieties, we can express this as follows: the strata are the loci where the relative position of the chain of de Rham cohomology groups $H^1_{DR}(A_i)$ and the chain of Hodge filtrations inside each $H^1_{DR}(A_i)$ is constant. This relative position "is" an element of the extended affine Weyl group $\widetilde{W}$ of the group GSp_{2g}, and the so-called admissible set $\mathrm{Adm} \subset \widetilde{W}$ is the (finite) set of relative positions which actually occur. The KR stratification on the space $\mathcal{A}_g$ consists of only one stratum, and hence does not provide any interesting information.

In general neither of these stratifications is a refinement of the other one. Nevertheless, there are some relations between them. For instance, the ordinary Newton stratum (which is open and dense in $\mathcal{A}_I$) is precisely the union of the maximal KR strata. At the other extreme, the supersingular locus is not in general a union of KR strata. However, it is our impression that those KR strata which are entirely contained in the supersingular locus make up a significant part of it. We call these KR strata *supersingular*.

We can produce a list of supersingular KR strata which admit a very simple geometric description in terms of Deligne-Lusztig varieties. To give a more precise description, we identify the elements of $\widetilde{W}$, and in particular of Adm with alcoves in the standard apartment (in the Bruhat-Tits building of GSp_{2g}). Each alcove x is determined by its vertices $x_0, \dots, x_g$. In particular we have the base alcove $\tau = (\tau_0, \dots, \tau_g)$, which corresponds to the unique 0-dimensional KR stratum. Fix $0 \leq i \leq [g/2]$. Let

$$W_{\{i,g-i\}} = \{x \in \mathrm{Adm};\ x_i = \tau_i, x_{g-i} = \tau_{g-i}\}.$$

It is not hard to show that for $x \in W_{\{i,g-i\}}$, the KR stratum $\mathcal{A}_x$ is contained in the supersingular locus. We conjecture that the set $\bigcup_i W_{\{i,g-i\}}$ is the set of supersingular KR strata.

We denote by $G_{\{i,g-i\}}$ the algebraic group over $\mathbb{F}_p$ whose Dynkin diagram is obtained by removing the vertices i and $g-i$ from the extended Dynkin diagram of GSp_{2g}, which splits over $\mathbb{F}_{p^2}$, and where the Frobenius (of $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$) acts by $j \mapsto g-j$. The set $W_{\{i,g-i\}}$ can be identified with the Weyl group of $G_{\{i,g-i\}}$ in a natural way.

Theorem 1. *Fix a point $(A_i)_i$ in the minimal KR stratum $\mathcal{A}_\tau$. Let $0 \leq i \leq [g/2]$, and let $x \in W_{\{i,g-i\}}$. Denote by π the projection from $\mathcal{A}_{g,I}$ to the analogous moduli space of partial lattice chains $(B_i \to B_{g-i})$. There is an isomorphism*

$$\pi^{-1}((A_j)_{j\in\{i,g-i\}}) \xrightarrow{\cong} \mathrm{Flag}(G_{\{i,g-i\}}).$$

Theorem 2. *Let $0 \leq i \leq [g/2]$, and let $x \in W_{\{i,g-i\}}$. We identify $W_{\{i,g-i\}}$ with the Weyl group of $G_{\{i,g-i\}}$, and let x^{-1} be the inverse element of x in this Weyl group. The KR stratum $\mathcal{A}_x$ is a disjoint union*

$$\mathcal{A}_x \xrightarrow{\cong} \coprod X(x^{-1}),$$

of copies of $X(x^{-1})$, which is by definition the Deligne-Lusztig variety for $G_{\{i,g-i\}}$ associated with x^{-1}. The union ranges over the finite set $\pi(\mathcal{A}_\tau)$, where π is as in the previous theorem.

As a direct corollary, we obtain

Corollary 1.
(1) *If $p \geq 2g$, and $w \in W_{\{i,g-i\}}$ for some $0 \leq i \leq [g/2]$, then the KR stratum associated with w is affine.*
(2) *There is an explicit formula for the number of connected components of KR strata as above (see [1], section 6).*

(3) *The dimension of the supersingular locus is greater or equal to $g^2/2$ if g is even, and $g(g-1)/2$ is g is odd.*

Although the union of the supersingular KR strata is not all of the supersingular locus, we still get a significant part. The following table backs this up for small g. The dimension of the whole moduli space $\mathcal{A}_I$ is $g(g+1)/2$. The dimension of the union of all superspecial KR strata is $g^2/2$ if g is even, and $g(g-1)/2$ otherwise (and this is how part (3) of the corollary is obtained). The numbers of KR strata, and of KR strata of p-rank 0 can be obtained from Haines' paper [2], Prop. 8.2, together with the results of Ngô and Genestier [4]; we indicate it to show the combinatorial complexity of these questions. The dimension of the p-rank 0 locus was obtained by a computer program. We conjecture that it is given by $[g^2/2]$ in general. This formula has been checked for all $g \leq 9$. If the conjecture holds true, it follows in particular that for g even, the dimension of the supresingular locus is $g^2/2$. Note that for $g = 5$ we do not know the dimension of the supersingular locus; for $g = 6$ we know it only because it has to lie between the dimension of the union of all superspecial KR strata and the dimension of the p-rank 0 locus. As a word of warning one should say that neither of these loci is equi-dimensional in general.

g	1	2	3	4	5	6
number of KR strata	3	13	79	633	6331	75973
number of KR strata of p-rank 0	1	5	29	233	2329	27949
dim. of union of superspecial KR strata	0	2	3	8	10	18
dim. of supersingular locus	0	2	3	8	?	18
dim. of p-rank 0 locus	0	2	4	8	12	18
$\dim \mathcal{A}_I$	1	3	6	10	15	21

Furthermore, it can be shown that any irreducible component of maximal dimension of the union of all superspecial KR strata is actually an irreducible component of the p-rank 0 locus, and hence in particular an irreducible component of the supersingular locus.

For further details we refer to our recent preprint [1].

References

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