Affine Deligne Lusztig varieties

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Let $\mathbb{F}_q$ be a finite field, $k \supset \mathbb{F}_q$ an algebraic closure, $L = k((\epsilon))$ the field of Laurent series, and $\mathfrak{o} = k[[\epsilon]]$ the ring of power series. Let G be a split reductive group over $\mathbb{F}_q$. To simplify the exposition below, we assume that G is semsisimple and simply connected. We denote by σ the Frobenius on k, L (acting by $\sum_i a_i\epsilon^i \mapsto \sum_i a_i^q\epsilon^i$), $G(k)$ and $G(L)$. We fix a split maximal torus and a Borel subgroup $T \subset B \subset G$, and denote by $I \subset G(\mathfrak{o})$ the standard Iwahori subgroup. We have the Bruhat-Iwahori decomposition

$$G(L) = \coprod_{w\in W_a} IwI,$$

the union ranging over the affine Weyl group W_a of G.

Definition 1. *Let $b \in G(L)$, and $x \in W_a$. The affine Deligne-Lusztig variety associated to b and x is*

$$X_x(b) = \{g \in G(L)/I;\ g^{-1}b\sigma(g) \in IxI\}.$$

If we identify $G(L)/I$ with the set of alcoves in the Bruhat-Tits building of the group G over L, then we can express the definition as follows: $X_x(b)$ is the set of all alcoves C such that the Weyl distance between C and its "Frobenius translate" $b\sigma C$ (for the twisted Frobenius automorphism $b\sigma$) is the fixed element w. We can view the quotient $G(L)/I$ as the (set of k-valued points of) the affine flag variety for G, an ind-scheme over k. Then $X_x(b)$ is a locally closed subset and is a k-scheme (with the reduced scheme structure) which is locally of finite type over k. In general, $X_x(b)$ will have infinitely many irreducible components.

The definition of affine Deligne-Lusztig varieties is obviously analogous to the definition of usual Deligne-Lusztig varieties [1], which live in the finite-dimensional flag variety G/B. There are however a number of important differences. In the finite-dimensional case, the parameter b is irrelevant, and one therefore usually sets $b = 1$. This is a consequence of Lang's theorem. Moreover, the Deligne-Lusztig variety associated to an element w of the finite Weyl group W is smooth and equi-dimensional of dimension $\ell(w)$, the length of w. On the other hand, for many pairs (b, x) the affine Deligne-Lusztig variety $X_x(b)$ is in fact empty; it is a difficult problem to decide which affine Deligne-Lusztig varieties are non-empty and to determine their dimensions. This is the question we focus on in the sequel.

We mention that there is an obvious variant where I is replaced by the "maximal compact" group $G(\mathfrak{o})$. This case is in fact much better understood than the Iwahori case; for instance there is a relatively simple criterion determining which affine Deligne-Lusztig varieties are non-empty, and a formula for their dimension. This formula was conjectured by Rapoport [3]; in [2] the conjecture was reduced to the so-called superbasic case, and this case was treated by Viehmann [5].

Apart from being interesting in their own right, affine Deligne-Lusztig varieties are important for studying the reduction of Shimura varieties in positive characteristic. For details, see [2], in particular section 5.10.

As an example, let us discuss the case of $G = SL_2$, $b = 1$. Let C be an alcove in the Bruhat-Tits tree. The Frobenius automorphism preserves the distance of the alcove to the rational building (i. e. the building over $\mathbb{F}_q$, the fixed point set of σ). If C lies in the rational building, then $\sigma C = C$, so the Weyl distance we get is the identity element in W_a. If the distance of C to the rational building is $d > 0$, then the distance of σC to the rational building is d, as well, and it is easy to see that as a consequence (because alcoves outside the rational building must be moved by σ) the distance from C to σC is $2d - 1$. This implies that their Weyl distance has odd length. There are two elements of any given odd length in W_a. However it is not hard to see that both these elements can be obtained as the Weyl distance of C and σC for some C. So the result in this case is that $X_x(1)$ is non-empty if and only if $x = \mathrm{id}$ or $\ell(x)$ is odd. In the general case, it is much harder to describe the result in such a succinct way.

Nevertheless, using results about σ-conjugacy classes in unipotent groups, one can prove the following criterion which reduces the question of non-emptiness to a question about orbit intersections in the group, or the affine flag variety. Denote by $U \subset B$ the unipotent radical. For an element w of the finite Weyl group W, write ${}^wU := wUw^{-1}$.

Proposition 2. *Let $b = \epsilon^\nu$ be a translation element. Then $X_x(b) \neq \emptyset$ if and only if there exists $w \in W$ such that*

$$ {}^wU(L)\epsilon^{w\nu}I \cap IxI \neq \emptyset. $$

This criterion can be reformulated as a purely combinatorial method to determine non-emptiness of affine Deligne-Lusztig varieties by analyzing the images of end points of galleries in the building of a fixed type, under retractions "from infinity" with respect to finite Weyl chambers. With some more effort, one can also determine the dimensions of non-empty affine Deligne-Lusztig varieties in this way. See [2]. Furthermore, the criterion can be generalized to the case of general b; the group U then has to be replaced by a group of the form $(I \cap M)N$, where P is a parabolic subgroup with Levi decomposition $P = MN$. This generalization will be discussed in a forthcoming paper. The criterion can be used to produce examples (with a computer program); see [2].

We have the following conjecture, which is an extended version of the conjecture stated in [4]. We consider two maps from W_a to W. The map η_1 is just the projection from W_a to W. To describe the second map, we identify W with the set of Weyl chambers. We define $\eta_2(x) = w$, where w is the unique element in W such that $w^{-1}x\mathbf{a}$ is contained in the dominant chamber, where $\mathbf{a}$ is the base alcove corresponding to I. We say that $x \in W_a$ lies in the shrunken Weyl chambers, if $U_\alpha \cap {}^xI \neq U_\alpha \cap I$ for all finite roots α (where U_α denotes the corresponding root subgroup). Furthermore, let S denote the set of (finite) simple reflections.

Conjecture 3. *a) Let $x \in W_a$ be an element of the shrunken Weyl chambers. Then $X_x(1) \neq \emptyset$ if and only if*

$$\eta_2(x)^{-1}\eta_1(x)\eta_2(x) \in W \setminus \bigcup_{T \subsetneq S} W_T,$$

and in this case

$$\dim X_x(1) = \frac{1}{2}\left(\ell(x) + \ell(\eta_2(x)^{-1}\eta_1(x)\eta_2(x))\right)$$

b) If $[b]$ is an arbitrary σ-conjugacy class, then there exists $n_0 \in \mathbb{Z}_{\geq 0}$, such that for all $x \in W_a$ of length $\ell(x) \geq n_0$, we have

$$X_x(b) \neq \emptyset \Longleftrightarrow X_x(1) \neq \emptyset.$$

There is a large amount of evidence for the conjecture, provided by a computer program which relies on the combinatorial criterion stated above. For groups of type A_2, C_2 part a) of the conjecture was proved by Reuman [4]. We can prove one direction of part a) of the conjecture in general. In fact, for this direction the assumption that x lies in the shrunken Weyl chambers is not necessary. The converse direction is not true for all x, however.

Theorem 4. *Let $x \in W_a$, and write $x = \epsilon^\lambda w$, $w \in W$. Assume that $x \notin W$ and that $\eta_2(x)^{-1}\eta_1(x)\eta_2(x) \in \bigcup_{T \subsetneq S} W_T$. Then $X_x(1) = \emptyset$.*

Note that if x is contained in the finite Weyl group W, then clearly $X_x(1) \neq \emptyset$ (it contains the corresponding finite Deligne-Lusztig variety).

In our proof of the theorem, the key point is that we exhibit a criterion which implies that for certain $x \in W_a$ the map

$$I \times I_M x I_M \longrightarrow IxI, \quad (i, m) \mapsto i^{-1} m \sigma(i),$$

is surjective. Here M is a Levi subgroup of G which contains x, and $I_M = I \cap M$. For $x = \mathrm{id}$, $M = A$, this just says that I is a single σ-conjugacy class, which is well known. It is clear that if x has this property, then there are strong restrictions on $X_x(b)$ being non-empty, and a careful analysis of the situation yields the theorem. Details will be published in a forthcoming paper.

References

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