Reduction of Shimura varieties and Deligne-Lusztig varieties

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We discuss some analogies, and direct relations, between Deligne-Lusztig varieties and Kottwitz-Rapoport strata in the reduction of Siegel modular varieties with Iwahori level structure.

1. Deligne-Lusztig varieties

Let G_0 be a connected reductive group over $\mathbb{F}_q$, the finite field with q elements. Fix a maximal torus and a Borel subgroup $T_0 \subset B_0$ in G_0, both defined over $\mathbb{F}_q$. Denote by k an algebraic closure of $\mathbb{F}_q$, and by G, B, T the base change to k. Let W be the absolute Weyl group, and σ the Frobenius automorphism. In [2], Deligne and Lusztig defined the following locally closed subvarieties of G/B, which nowadays are called Deligne-Lusztig varieties:

$$X(w) = \{gB \in G/B;\ g^{-1}\sigma(g) \in BwB\}.$$

We have the following "local model diagram":

$$\begin{array}{ccccccc}
G/B & \xleftarrow{\text{proj.}} & G & \xrightarrow{L} & G & \xrightarrow{\text{proj.}} & G/B \\
\uparrow & & \uparrow & & & & \uparrow \\
X(w) & \longleftarrow & \widetilde{X(w)} & & \longrightarrow & & C(w)
\end{array}$$

where L is the Lang map $g \mapsto g^{-1}\sigma(g)$, $C(w) = BwB/B$ denotes the Schubert cell attached to w, and $\widetilde{X(w)}$ is equal to the inverse image of $X(w)$ under the projection, and at the same time to the inverse image of $C(w)$ under the composition $\text{proj.} \circ L$. All horizontal arrows in this diagram are smooth and surjective, and we obtain a similar diagram by replacing $X(w)$ and $C(w)$ by their closures in G/B (and replacing $\widetilde{X(w)}$ accordingly). Therefore we obtain as a direct corollary:

Proposition 1. *The Deligne-Lusztig variety $X(w)$ is smooth. Its dimension is $\dim X(w) = \dim C(w) = \ell(w)$, the length of w. Its closure $\overline{X(w)}$ is smoothly equivalent to the Schubert variety $\overline{C(w)}$, and in particular is normal and Cohen-Macaulay. We have*

$$\overline{X(w)} = \bigsqcup_{v \le w} X(v) \qquad \textit{(as sets)},$$

where $\le$ denotes the Bruhat order on W.

Haastert showed that all Deligne-Lusztig varieties are quasi-affine. Furthermore, we note the following result (see [4]):

Proposition 2. *Let S' be a subset of the set S of simple reflections such that S' meets every σ-orbit. Then*

$$\bigcup_{s \in S'} X(s) \cup X(1)$$

is connected.

The proposition is a refinement of Lusztig's criterion for the connectedness criterion which says that $X(w)$ is connected if and only if $S' := \{s \in S;\ s \leq w\}$ satisfies the condition of the proposition.

2. Reduction of Shimura varieties

Fix an integer $g \leq 1$ and a prime number p. We consider the moduli space $\mathcal{A}$ of g-dimensional principally polarized abelian varieties over k, an algebraic closure of $\mathbb{F}_p$. It is a smooth k-variety, and we use a level structure away from p such that it is connected. Furthermore, we consider the moduli space $\mathcal{A}_I$ of abelian varieties with Iwahori level structure at p, which parametrizes chains

$$A_0 \to A_1 \to \cdots \to A_g$$

of isogenies of order p of g-dimensional abelian varieties with principal polarizations λ_0, λ_g on A_0, A_g, such that the pull-back of λ_g to A_0 is $p\lambda_0$.

Denote by Λ_i the following chain of k-vector spaces: $\Lambda_i = k^{2g}$, $i = 0, \dots, g$, and we fix maps $\alpha_i := \mathrm{diag}(0, \dots, 0, 1, 0, \dots, 0)\colon \Lambda_i \to \Lambda_{i+1}$, where the 1 is in the $(i+1)$-th position, $i = 0, \dots, g-1$. We equip Λ_0 and Λ_g with the standard symplectic pairing.

We recall the definition of the "local model" à la de Jong [1], Deligne/Pappas and Rapoport/Zink:

$$M^{\mathrm{loc}}(S) = \{(\mathcal{F}_i)_i \in \prod_{i=0}^{g} \mathrm{Grass}_g(\Lambda_i)(S);\ \alpha_i(\mathcal{F}_i) \subseteq \mathcal{F}_{i+1}, \\ \mathcal{F}_0, \mathcal{F}_g \text{ totally isotropic}\}.$$

The local model can be identified with a union of Schubert varieties in the affine flag variety for the group GSp_{2g}. and the corresponding local model diagram:

$$\begin{array}{ccccc} \mathcal{A}_I & \longleftarrow & \widetilde{\mathcal{A}}_I & \longrightarrow & M^{\mathrm{loc}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{A}_w & \longleftarrow & \widetilde{\mathcal{A}}_w & \longrightarrow & C(w) \end{array}$$

Here $\widetilde{\mathcal{A}}_I$ is the space of pairs $((A_\bullet)_\bullet, \Psi)$, where $(A_\bullet)_\bullet \in \mathcal{A}_I$ and Ψ is an isomorphism of chains $H^1_{DR}(A_\bullet/S) \xrightarrow{\sim} \Lambda_\bullet \otimes \mathcal{O}_S$. The morphism $\widetilde{\mathcal{A}}_I \to \mathcal{A}_I$ is a torsor for the automorphism group scheme of the chain $\Lambda_\bullet$ which is smooth, and using the theory of Grothendieck and Messing, one can show that the morphism $\widetilde{\mathcal{A}}_I \to M^{\mathrm{loc}}$ is smooth, too. In the lower row of the diagram, w is an element of the extended affine Weyl group such that the corresponding Schubert cell $C(w)$ lies in M^{loc}, and $\widetilde{\mathcal{A}}_w$ is the inverse image of $C(w)$ in $\widetilde{\mathcal{A}}_I$.

The locally closed subvarieties $\mathcal{A}_w \subset \mathcal{A}_I$ are defined by the above diagram; they are called Kottwitz-Rapoport (KR) strata, and by the definition the local structure of these strata and their closures is the same as the structure of the corresponding Schubert cells and Schubert varieties in the affine flag variety.

Using that the Hodge bundle on $\mathcal{A}_g$ is ample, one can show that all KR strata are quasi-affine, [7] Theorem 5.4. It turns out that KR strata are usually connected:

Theorem 1 ([7] Theorem 7.4, Corollary 7.5). *Every KR-stratum which is not contained in the supersingular locus of $\mathcal{A}_I$ is connected.*

One of the ingredients of the proof is the above mentioned result on Deligne-Lusztig varieties (in the special case of certain unitary groups, where it was first proved by Ekedahl and van der Geer). The structure of those KR strata that are contained in the supersingular locus can be made very explicit. We have

Theorem 2 ([6] §6, [7] Corollary 7.5). *Let $\mathcal{A}_w$ be a KR stratum which is contained in the supersingular locus. Then there exists $0 \leq i \leq \frac{g}{2}$ such that for every point $(A_\bullet)_\bullet \in k$, the abelian varieties A_i and A_{g-i} are superspecial. Furthermore $\mathcal{A}_w$ is the union of copies of Deligne-Lusztig varieties for a (non-split) group whose Dynkin diagram is obtained from the extended Dynkin diagram of type $\tilde{C}_2$ by omitting the vertices i and $g - i$.*

As a corollary of these results and a computation of the dimension of the p-rank 0 locus in $\mathcal{A}_I$, one obtains

Corollary 1 ([7] Theorem 1.1). *If g is even, then the supersingular locus in $\mathcal{A}_I$ has dimension $g^2/2$. If g is odd, then its dimension lies between $g(g-1)/2$ and $(g+1)(g-1)/2$.*

This should be compared with the result of Li and Oort [8] that the dimension of the supersingular locus in $\mathcal{A}_g$ is $[g^2/4]$. Note also that th supersingular locus in $\mathcal{A}_I$ is not equidimensional as soon as $g \geq 2$. In joint work with Maarten Hoeve we investigated the relationship between KR strata (also in the general parahoric case) to the Ekedahl-Oort stratification, see [5]. Compare also the work of Ekedahl and van der Geer [3].

References

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