Stratifications of affine Deligne-Lusztig varieties

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UNIVERSITÄT DUISBURG ESSEN

Classical Deligne-Lusztig varieties

G_0 connected reductive group / finite field $\mathbb{F}_q$, $T_0 \subset B_0 \subset G_0$.
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Fix $w \in W$. We set $X_w = \{g \in G/B;\ g^{-1}\sigma(g) \in BwB\}$.

Properties

- locally closed in G/B,
- smooth of dimension $\ell(w)$,
- $G_0(\mathbb{F}_q)$ acts on X_w, hence on $H^*(X_w, \mathbb{Q}_\ell)$.

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('positive') affine flag variety

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Relative position map:

$$\begin{aligned} \mathsf{inv}\colon \breve{G}/\breve{\mathcal{I}} \times \breve{G}/\breve{\mathcal{I}} &\longrightarrow \breve{\mathcal{I}}\backslash\breve{G}/\breve{\mathcal{I}} \cong \tilde{W} \\ (g,h) &\longmapsto g^{-1}h \end{aligned}$$

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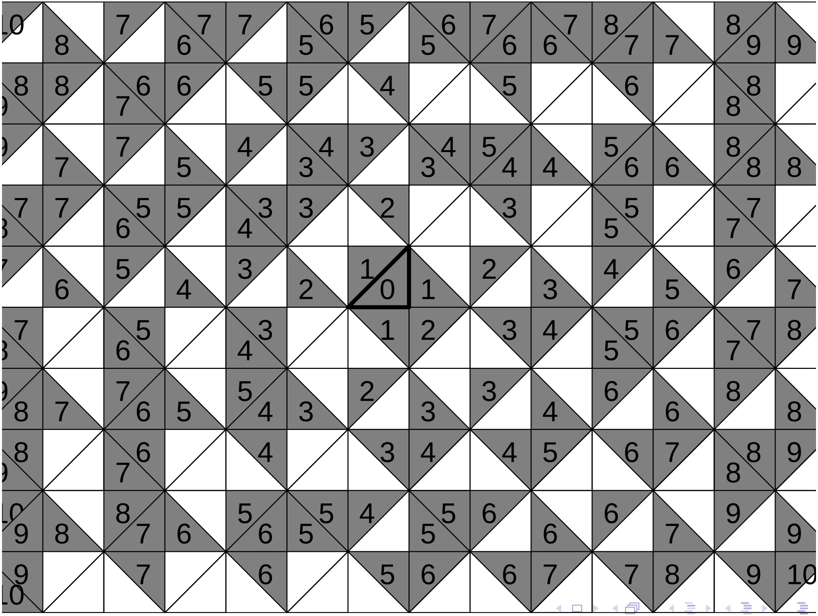
Example (SL_2, $b = 1$)

$$X_w(1) \neq \emptyset \quad \iff \quad w = \mathrm{id} \ \text{ or } \ \ell(w) \text{ odd}$$

Example: GSp_4, $b = \tau \neq \mathrm{id}$, $\ell(\tau) = 0$



The admissible set

Fix $t^\mu \in \tilde{W}$ translation element.

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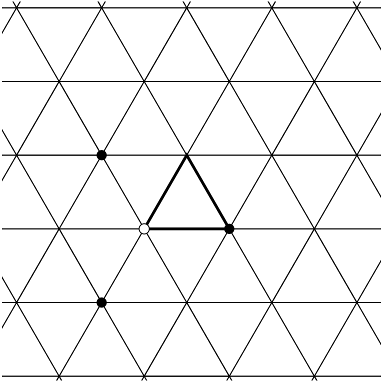

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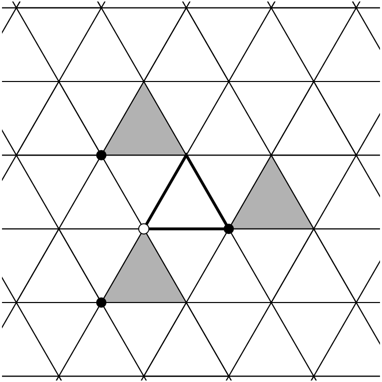

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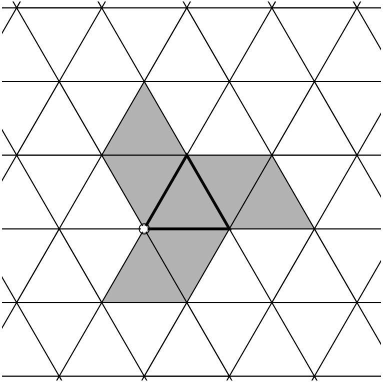

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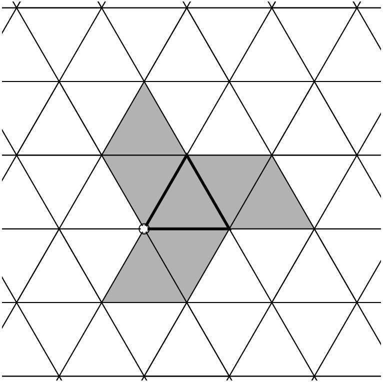

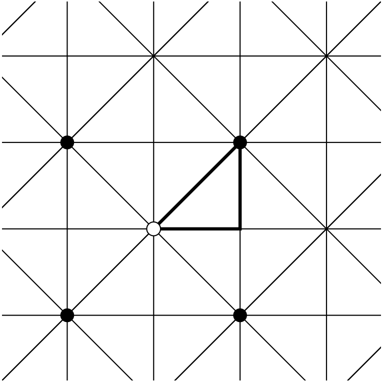

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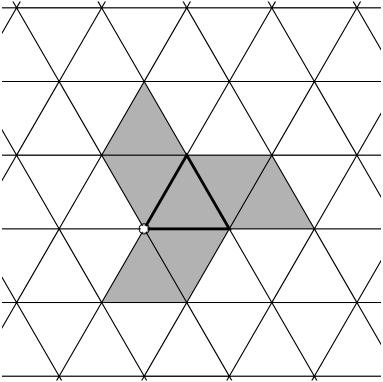

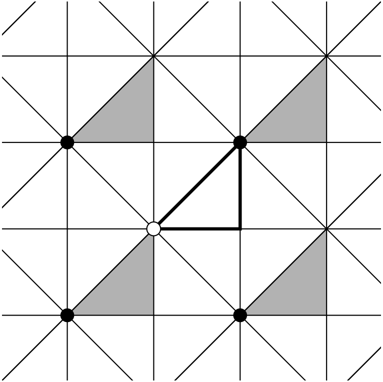

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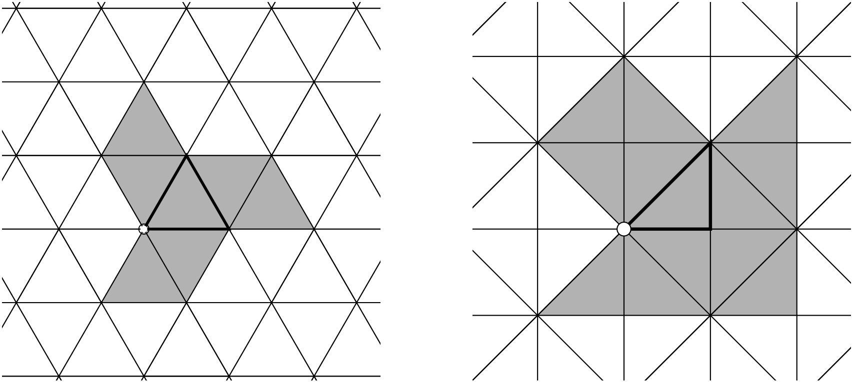

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- $X_w(b)$, $X(\mu, b)$ depend only on σ-conjugacy class $[b]$ of b.
- Can choose b in $\tilde{W}$.
- Given μ, $X(\mu, \tau) \neq \emptyset$ for a unique *length* 0 element $\tau \in \tilde{W}$.

dim $X(\mu, \tau) = \ ?$

Say $\boldsymbol{G} = GSp_{2g}$, $\mu = \omega_g^\vee$. Then $\dim X(\mu, \tau)$ equals the dimension of the supersingular locus of the moduli space of g-dimensional principally polarized abelian varieties with Iwahori level structure at p, over $\overline{\mathbb{F}}_p$.
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Theorem (G–Yu)

For g even, $\quad \dim X(\mu, \tau) = g^2/2.$

For g odd, $\quad g(g-1)/2 \le \dim X(\mu, \tau) \le (g+1)(g-1)/2.$

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Bonan: For $g \leq 5$ odd, $g(g-1)/2 = \dim X(\mu, \tau)$.

NB: Usually not equi-dimensional.

The $\mathbb{J}$-stratification

Relative position (for $K \subset \tilde{\mathbb{S}} \leftrightsquigarrow \breve{\mathcal{K}} \subset \breve{G}$)

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Definition (Chen–Viehmann)

$x, y \in \breve{G}/\breve{\mathcal{K}}$ lie in the same stratum $\iff$ for all $j \in \mathbb{J}$: $\mathrm{inv}_K(j, x) = \mathrm{inv}_K(j, y)$.

Intersecting with $X(\mu, b)_K$, get $\mathbb{J}$-stratification on $X(\mu, b)_K$.

Finiteness properties

Theorem

The $\mathbb{J}$-strata in $\breve{G}/\breve{K}$ are locally closed.

Proposition ("Generalized gate property")

Let S be a bounded set of alcoves in $\mathcal{B}(\breve{G})$. There exists a finite set J' of alcoves in $\mathcal{B}(\mathbb{J})$ with the following property:

> *for every alcove $\mathfrak{j}$ in $\mathcal{B}(\mathbb{J})$ there exists an alcove $\mathfrak{j}' \in J'$ such that every alcove in S can be reached from $\mathfrak{j}$ via a minimal gallery passing through $\mathfrak{j}'$.*

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Proposition (Gate property)

Let $\mathfrak{r}$ be a simplex in the building, and $\mathcal{R}$ the set of all alcoves whose closure contains $\mathfrak{r}$.

For every alcove $\mathfrak{b}$ there exists an alcove $\mathfrak{g}$ in $\mathcal{R}$ such that

> *every alcove in $\mathcal{R}$ can be reached from $\mathfrak{b}$ via a minimal gallery passing through $\mathfrak{g}$.*

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The fully Hodge-Newton decomposable case

joint with Xuhua He, Sian Nie

$B(\mathbf{G}, \mu) = \{[b];\ X(\mu, b) \neq \emptyset\}$,

$\tau \in \tilde{W}$, $\ell(\tau) = 0$, such that $[\tau] \in B(\mathbf{G}, \mu)$.

Theorem (G–He–Nie)

The following conditions are equivalent:

1. *The pair* $(\mathbf{G}, \{\mu\})$ *is fully Hodge-Newton decomposable.*
2. *The coweight* μ *is minute:*

   if G split: $\langle \mu, \omega_i \rangle \leq 1$ *for all* i

3. *For any* $[b] \neq [\tau]$ *in* $B(\mathbf{G}, \{\mu\})$, dim $X(\mu, b)_K = 0$.
4. *"Bruhat–Tits stratification:" The space* $X(\mu, \tau)_K$ *is naturally a union of classical Deligne-Lusztig varieties.*

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The Bruhat-Tits stratification

In situation of the theorem, (4) means:

$$X(\mu,\tau)_K = \bigsqcup_{w \in \mathrm{Adm}(\mu) \cap {}^K\tilde{W}} \pi_K(X_w(\tau)),$$

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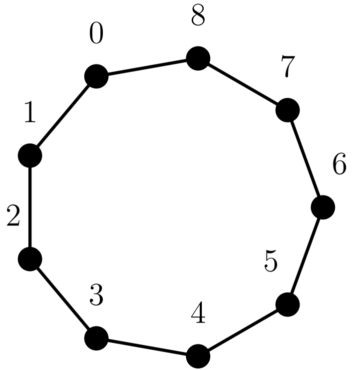
$$\pi_K(X_w(\tau)) = \bigsqcup_{j \in \mathbb{J}/\mathbb{J} \cap \breve{\mathcal{P}}'_w} jY(w),$$

where $Y(w) \subset \breve{\mathcal{P}}_w/\breve{\mathcal{I}}$ a classical DL variety.

Example: Unramified unitary group

G a quasi-split unitary group for unramified quadratic extension

$\leftrightsquigarrow$ Dynkin diagram $\tilde{A}_{n-1}$ with $\sigma(0) = 0$, $\sigma(i) = n - i$.



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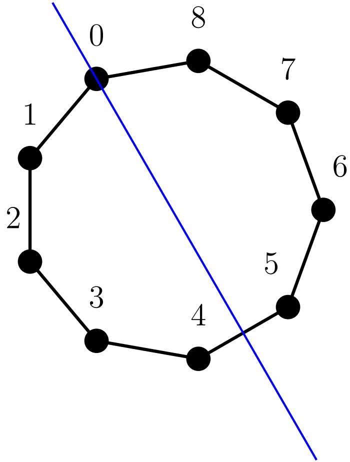

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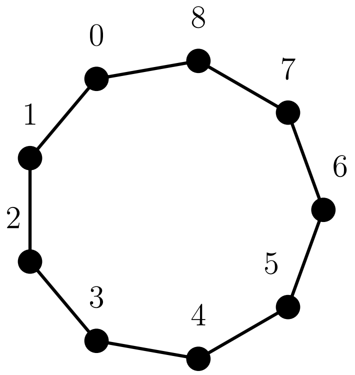

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$\mathrm{Int}(\tau)$ acts by rotation $i \mapsto i + 1$.

$\mathrm{Int}(\tau) \circ \sigma$ acts by reflection $0 \leftrightarrow 1$, $n - 1 \leftrightarrow 2$, ...

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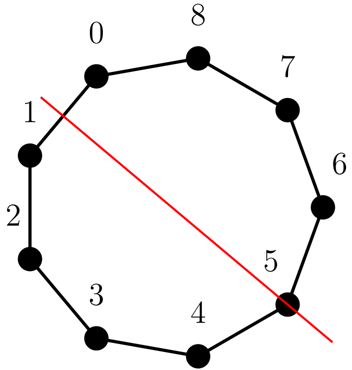

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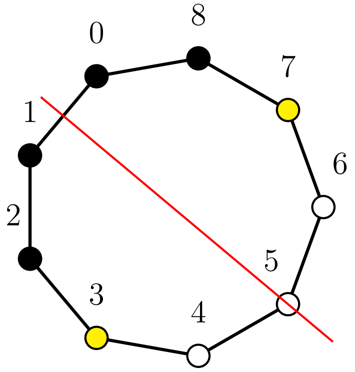

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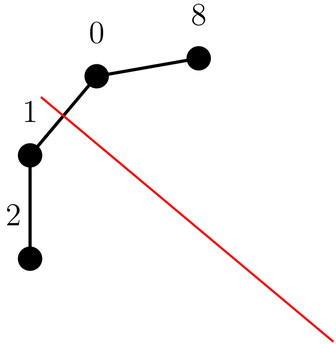

$\mu = \omega_1^\vee$

$w = s_0 s_8 \tau$

$Y(w)$ is a Deligne–Lusztig variety in a unitary group.

Theorem (G–He–Nie)

Assume that **G** *is quasi-simple over* $\breve{F}$ *and* $\mu \neq 0$*. Then* $(\mathbf{G}, \{\mu\})$ *is fully Hodge-Newton decomposable if and only if the associated triple* (W_a, μ, σ) *is one of the following:*

$(\tilde{A}_{n-1}, \omega_1^\vee, \mathrm{id})$ $(\tilde{A}_{2m-1}, \omega_1^\vee, \tau_1\varsigma_0)$ $(\tilde{A}_3, \omega_2^\vee, \varsigma_0)$	$(\tilde{A}_{n-1}, \omega_1^\vee, \tau_1^{n-1})$ $(\tilde{A}_{n-1}, \omega_1^\vee + \omega_{n-1}^\vee, \mathrm{id})$ $(\tilde{A}_3, \omega_2^\vee, \tau_2)$	$(\tilde{A}_{n-1}, \omega_1^\vee, \varsigma_0)$ $(\tilde{A}_3, \omega_2^\vee, \mathrm{id})$
$(\tilde{B}_n, \omega_1^\vee, \mathrm{id})$	$(\tilde{B}_n, \omega_1^\vee, \tau_1)$	
$(\tilde{C}_n, \omega_1^\vee, \mathrm{id})$	$(\tilde{C}_2, \omega_2^\vee, \mathrm{id})$	$(\tilde{C}_2, \omega_2^\vee, \tau_2)$
$(\tilde{D}_n, \omega_1^\vee, \mathrm{id})$	$(\tilde{D}_n, \omega_1^\vee, \varsigma_0)$	

Comparison in the Coxeter case

Coxeter case (G–He)

Fully HN decomposable +

$K = \tilde{\mathbb{S}} \setminus \{v\}, \quad \sigma(K) = K$ +

for all $w \in \mathrm{Adm}(\mu) \cap {}^K\tilde{W}$ with $X_w(\tau) \neq \emptyset$, w is twisted Coxeter:

$$\mathrm{supp}(w) := \{s \in \tilde{\mathbb{S}};\ s \leq w\}$$

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Theorem (G)

In the Coxeter cases, the $\mathbb{J}$-stratification coincides with the Bruhat–Tits stratification.

$\mathrm{inv}_K(j, -)$ is constant on BT strata

Proposition (Gate property)

Let $\mathfrak{r}$ be a simplex in the building, and $\mathfrak{R}$ the set of all alcoves whose closure contains $\mathfrak{r}$.

For every alcove $\mathfrak{b}$ there exists an alcove $\mathfrak{g}$ in $\mathfrak{R}$ such that

> *every alcove in $\mathfrak{R}$ can be reached from $\mathfrak{b}$ via a minimal gallery passing through $\mathfrak{g}$.*

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Return to the setting of classical DL varieties.

Proposition (Lusztig)

Let $G_0/\mathbb{F}_q$, $w \in W$ twisted Coxeter, $g \in G_0(\mathbb{F}_q)$, $h \in X_w$. Then

$$\mathrm{inv}(g, h) = w_0, \qquad \textit{the longest element of } W.$$

Extremal cases

joint with Xuhua He, Michæl Rapoport

Assume that μ is not central in any simple factor of ***G*** over $\breve{F}$.

Theorem (Equi-maximal-dimensional case, G–H–R)

Then $X(\mu, \tau)_K$ *is equi-dimensional of dimension* $\langle \mu, 2\rho \rangle$

$\Longleftrightarrow$ (W_a, σ, μ, K) *is isomorphic to one of the following:*

1. $(\tilde{A}_{n-1}, \circlearrowleft_1, \omega_1^\vee, \emptyset)$ ← Drinfeld case
2. $(\tilde{A}_{n-1} \times \tilde{A}_{n-1}, \curvearrowleft\!\!\curvearrowright, (\omega_1^\vee, \omega_{n-1}^\vee), \emptyset)$
3. $(\tilde{A}_3, \circlearrowleft_2, \omega_2^\vee, \emptyset)$

Dimension 0

Theorem (G-He-Rapoport)

$\dim X(\mu, \tau)_K = 0 \iff$ *(W_a, σ, μ) is isomorphic to $(\tilde{A}_{n-1}, \mathrm{id}, \omega_1^\vee)$ for some n.*

Lubin–Tate case

Finite fibers

Fix a pair $K \subsetneq K'$ of F-rational parahoric level structures.

Have projection $\pi_{K,K'}\colon X(\mu,\tau)_K \to X(\mu,\tau)_{K'}$.

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Theorem (G–He–Rapoport)

Then

all fibers of $\pi_{K,K'}$ are finite $\iff$ *LT case or*

Dynkin type $\tilde{A}_{n-1}$ with $\sigma(0) = 0$, $\sigma(i) = n - i$, and $\mu = \omega_1^\vee$, and

- $K' \setminus K \subset \{s_0, s_{\frac{n}{2}}\}$*, and if* $s_i \in K' \setminus K$*, then* $s_{i+1} \notin K$.