# Affine Springer Fibers and Affine Deligne-Lusztig Varieties

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**Abstract.** We give a survey on the notion of affine Grassmannian, on affine Springer fibers and the purity conjecture of Goresky, Kottwitz, and MacPherson, and on affine Deligne-Lusztig varieties and results about their dimensions in the hyperspecial and Iwahori cases.

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## 1. Introduction

These notes are based on the lectures I gave at the Workshop on Affine Flag Manifolds and Principal Bundles which took place in Berlin in September 2008. There are three chapters, corresponding to the main topics of the course. The first one is the construction of the affine Grassmannian and the affine flag variety, which are the ambient spaces of the varieties considered afterwards. In the following chapter we look at affine Springer fibers. They were first investigated in 1988 by Kazhdan and Lusztig [41], and played a prominent role in the recent work about the "fundamental lemma", culminating in the proof of the latter by Ngô. See Section 3.8. Finally, we study affine Deligne-Lusztig varieties, a "$\sigma$-linear variant" of affine Springer fibers over fields of positive characteristic, $\sigma$ denoting the Frobenius automorphism. The term "affine Deligne-Lusztig variety" was coined by Rapoport who first considered the variety structure on these sets. The sets themselves appear implicitly already much earlier in the study of twisted orbital integrals.

We remark that the term "affine" in both cases is not related to the varieties in question being affine, but rather refers to the fact that these are notions defined in the context of an affine root system. We include short reminders about the corresponding non-affine notions, i.e., Springer fibers and Deligne-Lusztig varieties.

No originality for any of the results in this article is claimed; this is especially true for Chapter 3 where I am really only reporting about the work of others: Goresky, Kottwitz, Laumon, MacPherson, Ngô, ...

**1.1. Notation**

We collect some standard, mostly group-theoretic, notation which is used throughout this article. Let $k$ be a field (in large parts of Section 2 we can even work over an arbitrary commutative ring). In Section 3, $k$ is assumed to be algebraically closed. In Section 4, $k$ will be an algebraic closure of a finite field.

Let $\mathcal{O} = k[[\epsilon]]$ be the ring of formal power series over $k$, and let $L = k((\epsilon)) := k[[\epsilon]][\frac{1}{\epsilon}]$ be the field of Laurent series over $k$. In Section 4, we let $F = \mathbb{F}_q((\epsilon))$.

Let $G$ be a connected reductive group over $k$, or —in Section 4— over a finite field $\mathbb{F}_q$. We will assume that $G$ is split, i.e., that there exists a maximal torus $A \subseteq G$ which is isomorphic to a product of copies of the multiplicative group $\mathbb{G}_m$. We fix such a split torus $A$. We also fix a Borel subgroup $B$ of $G$ which contains $A$. We denote by $W$ the Weyl group of $A$, i.e., the quotient $N_G A/A$ of the normalizer of $A$ by $A$, and by $X_*(A) = \mathrm{Hom}(\mathbb{G}_m, A)$ the cocharacter lattice of $A$.

For notational convenience, we assume that the Dynkin diagram of $G$ is connected. We denote by $\Phi$ the set of roots given by the choice of $A$, and by $\Phi^+$ the set of positive roots distinguished by $B$. We let

$$X_*(A)_+ = \{\lambda \in X_*(A);\ \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}$$

denote the set of dominant cocharacters. An index $-_{\mathbb{Q}}$ denotes tensoring by $\mathbb{Q}$, e. g. we have $X_*(A)_{\mathbb{Q}}$, $X_*(A)_{\mathbb{Q},+}$. Similarly, we have $X_*(A)_{\mathbb{R}}$, etc. Let $\rho$ denote half the sum of the positive roots. For $\lambda, \mu \in X_*(A)$, we write $\lambda \leq \mu$ if $\mu - \lambda$ is a linear combination of simple coroots with non-negative coefficients.

Now we come to the "affine" situation. We embed $X_*(A)$ into $A(L) \subset G(L)$ by $\lambda \mapsto \lambda(\epsilon) =: \epsilon^\lambda$, where by $\lambda(\epsilon)$ we denote the image of $\epsilon$ under the map

$$L^\times = \mathbb{G}_m(L) \to A(L) \subset G(L)$$

induced by $\lambda$. The extended affine Weyl group (or Iwahori-Weyl group) $\widetilde{W}$ is defined as the quotient $N_{G(L)}T(L)/T(\mathcal{O})$. It can also be identified with the semi-direct product $W \ltimes X_*(A)$. On $\widetilde{W}$, we have a length function $\ell\colon \widetilde{W} \to \mathbb{Z}_{\geq 0}$,

$$\ell(w\epsilon^\lambda) = \sum_{\substack{\alpha>0 \\ w(\alpha)<0}} |\langle \alpha, \lambda \rangle + 1| + \sum_{\substack{\alpha>0 \\ w(\alpha)>0}} |\langle \alpha, \lambda \rangle|, \qquad w \in W, \lambda \in X_*(A)$$

Let $S \subset W$ denote the subset of simple reflections, and let $s_0 = \epsilon^{\tilde{\alpha}^\vee} s_{\tilde{\alpha}}$, where $\tilde{\alpha}$ is the unique highest root. The affine Weyl group $W_a$ is the subgroup of $\widetilde{W}$ generated by $S \cup \{s_0\}$. Then $(W_a, S \cup \{s_0\})$ is a Coxeter system, and the restriction of the length function is the length function on $W_a$ given by the fixed system of generators.

We can write the extended affine Weyl group as a semi-direct product $\widetilde{W} \cong W_a \rtimes \Omega$, where $\Omega \subset \widetilde{W}$ is the subgroup of length 0 elements. We extend the Bruhat order on $W_a$ to a partial order on $\widetilde{W}$, again called the Bruhat order, by setting $w\tau \le w'\tau'$ if and only if $\tau = \tau'$, $w \le w'$, for $w, w' \in W_a$, $\tau, \tau' \in \Omega$.

The subgroup $G(\mathcal{O})$ is a "hyperspecial" subgroup of $G(L)$, so we sometimes refer to a case relating to $G(\mathcal{O})$ as the hyperspecial case. We also consider the Iwahori subgroup $I \subset G(\mathcal{O})$ which we define as the inverse image of the opposite Borel $B^-(k)$ under the projection $G(\mathcal{O}) \to G(k)$, $\epsilon \mapsto 0$.

In the case $G = \mathrm{GL}_n$, we always let $A$ be the torus of diagonal matrices, and we choose the subgroup of upper triangular matrices $B$ as Borel subgroup. We can then identify the Weyl group $W$ with the subgroup of permutation matrices in $\mathrm{GL}_n$, and we can identify the extended affine Weyl group $\widetilde{W}$ with the subgroup of matrices with exactly one non-zero entry in each row and column, which is of the form $\epsilon^i$, $i \in \mathbb{Z}$. The subgroup $\Omega \subset \widetilde{W}$ is isomorphic to $\mathbb{Z}$. For $G = \mathrm{SL}_n$, we make analogous choices of the maximal torus and the Borel subgroup.

## 2. The Affine Grassmannian and the Affine Flag Manifold

We start by an introduction to the affine Grassmannian and affine flag variety of the group $G$. Both affine Springer fibers and affine Deligne-Lusztig varieties live inside one of these. As general references for the construction we name the papers by Beauville and Laszlo [4], Pappas and Rapoport [63] and Sorger [75]. Let $k$ be a field (at least in 2.1–2.4 we could work over any base ring, though).

### 2.1. Ind-Schemes

We first recall the notion of ind-scheme. Roughly speaking, an ind-scheme is just the union of an "ascending" system of schemes. To say precisely in which sense the union is taken, it is most appropriate to use the functorial point of view on schemes.

**Definition 2.1.** A *k-space* is a functor $F\colon (\mathrm{Sch})^o \to (\mathrm{Sets})$ which is a sheaf for the fpqc-topology, i.e., whenever $X = \bigcup_i U_i$ is a covering by (Zariski-)open subsets, then the sequence

$$F(X) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j),$$

is exact, and whenever $R \to R'$ is a faithfully flat homomorphism of $k$-algebras, then the sequence

$$F(\operatorname{Spec} R) \to F(\operatorname{Spec} R') \rightrightarrows F(\operatorname{Spec} R' \times_{\operatorname{Spec} R} \operatorname{Spec} R')$$

is exact. A morphism of $k$-spaces is a morphism of functors.

Here by exactness we mean that the map on the left hand side is injective, and that its image is equal to the subset of elements which have the same image under both maps on the right hand side. The first axiom is called the Zariski-sheaf axiom, for obvious reasons. The second condition is called the fpqc sheaf axiom, where fpqc stands for fidèlement plat quasi-compact.

**Remark 2.2.** 1. Because of the Zariski-sheaf axiom, one can equivalently consider (contravariant) functors from the category of *affine* schemes to the category of sets which satisfy the fpqc condition, or similarly (covariant) functors $G$ from the category of $k$-algebras to the category of sets for which $G(R) \to G(R') \rightrightarrows G(R' \otimes_R R')$ is exact for every faithfully flat homomorphism $R \to R'$.

2. Every $k$-scheme $Z$ gives rise to a $k$-space, also denoted by $Z$, by setting $Z(S) = \mathrm{Hom}_k(S, Z)$, the $S$-valued points of $Z$. The sheaf axiom for Zariski coverings is clearly satisfied, and Grothendieck's theory of faithfully flat descent shows that the fpqc sheaf axiom is satisfied as well.
3. The Yoneda lemma says that the functor from the category of schemes to the category of $k$-spaces is fully faithful. Those $k$-spaces which are in the essential image of this functor are called representable.

There is a standard method of "sheafification" in this context (see Artin's notes [2]). Therefore, many constructions available for usual sheaves can be carried out for $k$-spaces, as well. For instance, one shows similarly as for usual sheaves that inductive limits exist in the category of $k$-spaces, and that quotients of sheaves of abelian groups exist.

**Definition 2.3.** An *ind-scheme* is a $k$-space which is the inductive limit (in the category of $k$-spaces) of an inductive system of schemes, where the index set is the set $\mathbb{N}$ of natural numbers with its usual order, and where all the transition maps are closed immersions.

Of course, one could also allow more general index sets. Often one does not make the restriction that all transition maps must be closed immersions, and speaks of a *strict ind-scheme*, if this is the case. Since in the sequel this additional condition will always be satisfied, we omit the *strict* from the notation. See Drinfeld's paper [20] for generalities on ind-schemes and remarks about the relation to the notion of formal scheme.

We call an ind-scheme $X$ of *ind-finite type*, or *ind-projective*, etc., if we can write $X = \varinjlim X_n$ where each $X_n$ if of finite type, projective, etc. We call $X$ *reduced*, if it can be written as the inductive limit of a system $(X_n)_n$ where each $X_n$ is a reduced scheme. This is a somewhat subtle notion because usually there will be many ways to write a reduced ind-scheme as a limit of non-reduced schemes.

**Lemma 2.4.** *Let $X = \varinjlim_n X_n$ be an ind-scheme. Let $S \to X$ be a morphism from a quasi-compact scheme $S$ to $X$. Then there exists $n$ such that the morphism $S \to X$ factors through $X_n$.*

Note that this does not follow from the universal property of the inductive limit. Rather, the reason is that since $S$ is quasi-compact, the $S$-valued points of the sheafification of the inductive limit $\varinjlim X_n$ are just the $S$-valued points of the presheaf inductive limit, i.e., $X(S) = \varinjlim X_n(S)$ for quasi-compact $S$.

We will see many examples of ind-schemes (which are not schemes) below. A simple example which is quite helpful is the following: Suppose we want to view the power series ring $k[[\epsilon]]$ over the field $k$ as a $k$-scheme. This is easy because a power series is just given by its coefficients, so $k[[\epsilon]]$ is the set of $k$-valued points of the countably infinite product $\mathbb{A}^\infty = \prod_{i\geq 0} \mathbb{A}^1$, a perfectly reasonable scheme, even if it is infinite-dimensional. In fact, this scheme is just the spectrum of the polynomial ring in countably many variables. Similarly, we can express the set of doubly infinite power series $\sum_{i=-\infty}^{\infty} a_i\epsilon^i$ as an infinite product of affine lines. Now suppose we want to express the field $k((\epsilon))$ of Laurent series $\sum_i a_i\epsilon^i$ with $a_i = 0$ for all but finitely many $i < 0$ in a similar way. Clearly, it is contained in the product $\prod_{i=-\infty}^{\infty} \mathbb{A}^1$. But the condition that only finitely many coefficients with negative index may be non-zero cannot be expressed by polynomial equations! Therefore we cannot express $k((\epsilon))$ as a closed subscheme of the "doubly infinite" product. On the other hand, writing $k((\epsilon))$ as the union

$$k((\epsilon)) = \bigcup_{i\leq 0} \epsilon^i k[[\epsilon]],$$

we find an obvious ind-scheme structure on $k((\epsilon))$.

### 2.2. The Loop Group

We now fix a reductive linear algebraic group $G$ over $k$ (for most of this section, it is not important whether $G$ is reductive). The loop group $LG$ of $G$ is the $k$-space given by the following functor:

$$LG(R) = G(R((\epsilon))), \quad R \text{ a } k\text{-algebra.}$$

The terminology "loop group" refers to the fact that this construction is similar to the construction of the loop group in topology. There one considers the space of continuous maps from the circle $S^1$ to the given topological group. In the algebraic context, the circle is replaced by an infinitesimal pointed disc, i.e., the spectrum of $k((\epsilon))$. Similarly, we have the *positive loop group* $L^+G$, defined by

$$L^+G(R) = G(R[[\epsilon]]), \quad R \text{ a } k\text{-algebra.}$$

The positive loop group is actually an (infinite-dimensional) scheme. Let us first check this for $\mathrm{GL}_n$. The idea is to view $k[[\epsilon]] = \mathbb{A}^1_k(k[[\epsilon]])$ as an infinite product of affine lines over $k$, as explained above. Via the closed embedding

$$\mathrm{GL}_n \to \mathrm{Mat}_{n\times n} \times \mathrm{Mat}_{n\times n}, \qquad A \mapsto (A, A^{-1})$$

we identify $\mathrm{GL}_n(R[[\epsilon]])$ with the set of matrices

$$\{(A,B) \in \mathrm{Mat}_{n\times n}(R[[\epsilon]]) \times \mathrm{Mat}_{n\times n}(R[[\epsilon]]), \quad AB = 1\}.$$

Therefore $L^+G$ is the closed subscheme in $\prod_{i\geq 0}(\mathbb{A}^{n^2} \times \mathbb{A}^{n^2})$ given by the equations obtained from splitting the matrix equality $AB = 1$ into equations for each $\epsilon$-component. Given an arbitrary linear group $G$, we can embed $G$ as a closed subgroup into some $\mathrm{GL}_n$, and we see that $L^+G$ is a closed subscheme of $L^+\,\mathrm{GL}_n$.

**Definition 2.5.** The *affine Grassmannian* $\mathcal{G}rass_G$ *for* $G$ *is the quotient* $k$*-space* $LG/L^+G$.

The quotient in the category of $k$-spaces is the sheafification of the presheaf quotient $R \mapsto LG(R)/L^+G(R)$.

Similarly, if we choose a Borel subgroup $B^- \subseteq G$, then we have an Iwahori subgroup $\mathbf{I} \subseteq L^+G$, which by definition is the inverse image of $B^-$ under the projection $L^+G \to G$ (which maps $\epsilon$ to 0). Instead of the quotient of $LG$ by the positive loop group $L^+G$, we can consider the quotient by $\mathbf{I}$:

**Definition 2.6.** The *affine flag variety* $\mathcal{F}lag_G$ for $G$ is the quotient $k$-space $LG/\mathbf{I}$.

More generally, by taking quotients by "parahoric subgroups", we can define "partial affine flag varieties". We will show below that the affine Grassmannian and the affine flag variety are ind-schemes over $k$. In the case of $\mathrm{GL}_n$ (and with some more effort, of any classical group) these ind-schemes can be interpreted as parameter spaces of lattices or lattice chains (satisfying certain conditions).

**Remark 2.7.** The loop and positive loop constructions can be applied to any scheme over $k((\epsilon))$ and $k[[\epsilon]]$, resp. In particular, one can construct the loop group for a group $G$ over $k((\epsilon))$ which does not come from $k$ by base change. One obtains "twisted" loop groups, and their affine Grassmannians. This is worked out in the paper [63] by Pappas and Rapoport. The basic construction is the same; note however that the notion of parahoric subgroup is more subtle in general than in our case, see [34]. Certain properties from the non-twisted case are shown to carry over to the twisted case in loc. cit., but there are still many open questions.

### 2.3. Lattices

Let $k$ be a field, and let $R$ be a $k$-algebra. We denote by $R[[\epsilon]]$ the ring of formal power series over $R$, and by $R((\epsilon))$ the ring of Laurent series over $R$, i.e., the localization of $R[[\epsilon]]$ with respect to $\epsilon$. Let $r$, $n$ be positive integers. The $R[[\epsilon]]$-submodule $R[[\epsilon]]^n \subset R((\epsilon))^n$ is called the *standard lattice*, and is denoted by $\Lambda_R = \Lambda_{0,R}$.

**Definition 2.8.** 1. A *lattice* $\mathscr{L} \subset R((\epsilon))^n$ is a $R[[\epsilon]]$-submodule such that

(a) There exists $N \in \mathbb{Z}_{\geq 0}$ with

$$\epsilon^N \Lambda_R \subseteq \mathscr{L} \subseteq \epsilon^{-N}\Lambda_R, \text{ and}$$

(b) the quotient $\epsilon^{-N}\Lambda_R/\mathscr{L}$ is locally free of finite rank over $R$.

2. A lattice $\mathscr{L}$ is called *$r$-special*, if $\bigwedge^n \mathscr{L} = \epsilon^r \Lambda_R$.

We denote the set of all lattices in $R((\epsilon))^n$ by $\mathcal{L}att_n(R)$, and the set of all 0-special lattices by $\mathcal{L}att_n^0(R)$. Our goal is to equip these sets with an "algebraic-geometric structure". We will see below that $\mathcal{L}att_n$ and $\mathcal{L}att_n^0$ are ind-schemes and can be identified with the affine Grassmannian of $\mathrm{GL}_n$ and $\mathrm{SL}_n$ respectively.

We also define, for $N \geq 1$, subsets

$$\mathcal{L}att_n^{(N)}(R) \subset \mathcal{L}att_n(R), \qquad \mathcal{L}att_n^{0,(N)}(R) \subset \mathcal{L}att_n^0(R),$$

where the number $N$ in part (a) of the definition of a lattice is fixed. The functors $\mathcal{L}att_n^{(N)}(R)$, $\mathcal{L}att_n^{0,(N)}$ are projective schemes over $k$. Let us show this for $\mathcal{L}att_n^{0,(N)}$.

(Then $\mathcal{L}att_n^{(N)}(R)$ can be obtained as a disjoint union of schemes of the same form.) The morphism of functors

$$\mathcal{L}att_n^{0,(N)}(R) \to \mathrm{Grass}_N(\epsilon^{-N}\Lambda_k/\epsilon^N\Lambda_k)(R), \quad \mathscr{L} \mapsto \mathscr{L}/\epsilon^N\Lambda_R,$$

defines a closed embedding of $\mathcal{L}att_n^{0,(N)}$ into the Grassmann variety of $N$-dimensional subspaces of the ($2N$-dimensional) $k$-vector space $t^{-N}k[[\epsilon]]^n/t^N k[[\epsilon]]^n$. The image is the closed subscheme of all subspaces that are stable under the nilpotent endomorphism induced by $\epsilon$. We have

$$\mathcal{L}att_n(R) = \bigcup_N \mathcal{L}att_n^{(N)}(R), \qquad \mathcal{L}att_n^0(R) = \bigcup_N \mathcal{L}att_n^{0,(N)}(R).$$

We obtain ind-schemes

$$\mathcal{L}att_n = \bigcup \mathcal{L}att_n^{(N)}, \qquad \mathcal{L}att_n^0 = \bigcup \mathcal{L}att_n^{0,(N)}.$$

**Example 2.9.** The ind-scheme $\mathcal{L}att_1$ over $k$. The $k$-valued points of $\mathcal{L}att_1$ are the finitely generated $k[[\epsilon]]$-submodules of $k((\epsilon))$, i.e., the fractional ideals $(\epsilon^i)$, $i \in \mathbb{Z}$. So topologically, $\mathcal{L}att_1$ is simply the disjoint union of countably many points.

Let us determine the ind-scheme structure. An $R$-valued point $\mathscr{L} \in \mathcal{L}att_1(R)$, where $R$ is an arbitrary ring, is an $R[[\epsilon]]$-submodule $\mathscr{L} \subseteq R((\epsilon))$. The conditions that $\mathscr{L}$ is a lattice are equivalent to saying that $\mathscr{L}$ is generated, over $R[[\epsilon]]$, by an element of $R((\epsilon))^\times$ (cf. Lemma 2.11). This unit of $R((\epsilon))$ is determined by $\mathscr{L}$ up to multiplication by units of $R[[\epsilon]]$.

We have

$$R((\epsilon))^\times = \left\{ \sum_i a_i\epsilon^i \in R((\epsilon));\ \exists i_0 : a_{i_0} \in R^\times, a_j \text{ nilpotent for all } j < i_0 \right\},$$

so

$$R((\epsilon))^\times/R[[\epsilon]]^\times = \coprod_{i_0} \bigcup_{N\geq 1} \{(a_1,\dots,a_N);\ a_i \in R \text{ nilpotent}\}$$

Here for fixed $i_0$, the sets in the union over $N$ are embedded into each other in the obvious way (i.e., by extending tuples of smaller length by zeros).

This description shows that as an ind-scheme, $\mathcal{L}att_1$ is highly non-reduced. However, this phenomenon of non-reducedness will be of no importance for us. Note that nevertheless, $\mathcal{L}att_1$ is formally smooth, i.e., it satisfies the infinitesimal lifting criterion for smoothness.

### 2.4. The Affine Grassmannian for $\mathrm{GL}_n$

We prove that the functor $\mathcal{G}rass$ (which we defined above as the quotient of the loop group by the positive loop group) is representable by an ind-scheme in two steps: In this section, we deal with the case of $G = \mathrm{GL}_n$, and in the next section we consider the general case. In the case of the general linear group, we can describe $\mathcal{G}rass$ quite explicitly in terms of lattices. Every element $g \in LG(R)$ gives rise to a lattice $g\Lambda_R$. We obtain:

**Proposition 2.10.** *The affine Grassmannian for* $\mathrm{GL}_n$ *is isomorphic, as a* $k$*-space, to* $\mathcal{L}att_n$*. The affine Grassmannian for* $\mathrm{SL}_n$ *is isomorphic to* $\mathcal{L}att_n^0$*.*

To prove the proposition, one has to show that fpqc-locally on $R$, every lattice is free over $R[[\epsilon]]$: choosing a basis for a *free* lattice gives a representation by a matrix in $\mathrm{GL}_n(R((\epsilon)))$ which is well-defined up to an element of $\mathrm{GL}_n(R[[\epsilon]])$. It is enough to achieve this after a quasi-compact faithfully flat base change $R \to R'$, because the quotient $LG/L^+G$ is the *sheafification* of the presheaf quotient. Therefore the proposition follows from the equivalence of 1. and 4. of the following lemma:

**Lemma 2.11.** *Let* $\mathscr{L} \subset R((\epsilon))^n$ *be a* $R[[\epsilon]]$*-submodule. The following are equivalent:*

1. *The submodule* $\mathscr{L}$ *is a lattice.*
2. *The submodule* $\mathscr{L}$ *is a projective* $R[[\epsilon]]$*-module and* $\mathscr{L} \otimes_{R[[\epsilon]]} R((\epsilon)) = R((\epsilon))^n$*.*
3. *Zariski-locally on* $R$*,* $\mathscr{L}$ *is a free* $R[[\epsilon]]$*-module of rank* $n$ *(i.e., there exist* $f_1, \dots, f_r \in R$ *such that* $(f_1, \dots, f_r) = (1)$ *and for all* $i$*,* $\mathscr{L} \otimes_{R[[\epsilon]]} R_{f_i}[[\epsilon]]$ *is a free* $R_{f_i}[[\epsilon]]$*-module of rank* $n$*) and* $\mathscr{L} \otimes_{R[[\epsilon]]} R((\epsilon)) = R((\epsilon))^n$*.*
4. *fpqc-locally on* $R$*,* $\mathscr{L}$ *is a free* $R[[\epsilon]]$*-module of rank* $n$ *(i.e., there exists a faithfully flat ring homomorphism* $R \to R'$ *such that* $\mathscr{L} \otimes_{R[[\epsilon]]} R'[[\epsilon]]$ *is a free* $R'[[\epsilon]]$*-module) and* $\mathscr{L} \otimes_{R[[\epsilon]]} R((\epsilon)) = R((\epsilon))^n$*.*

Note that in 4., usually one has $R'[[\epsilon]] \neq R' \otimes_R R[[\epsilon]]$, and similarly in 3.

*Proof.* 1. $\Rightarrow$ 2. To simplify the notation, we assume that $\epsilon^N \Lambda_R \subseteq \mathscr{L} \subseteq \Lambda_R$. First note that for $s \in R$, we have

$$\Lambda_R \otimes_{R[[\epsilon]]} R[[\epsilon]]_s / \mathscr{L} \otimes_{R[[\epsilon]]} R[[\epsilon]]_s = (\Lambda_R/\mathscr{L}) \otimes_R R_s$$

as $R$-modules, so although $R_s[[\epsilon]]$ usually differs from $R[[\epsilon]]_s$, this difference does not matter for us. To prove 2., one shows that locally on $R$, there exists a basis $f_1, \dots, f_n$ of $\Lambda_R$ over $R[[\epsilon]]$ and $i_1, \dots i_n \in \mathbb{Z}_{\geq 0}$, such that the elements $\epsilon^j f_l$, $l = 1, \dots, n$, $j = i_l, \dots, N-1$, form an $R$-basis of $\mathscr{L}/\epsilon^N \Lambda_R$. This can be achieved by successively choosing suitable bases of the subquotients $\ker \epsilon^i / \ker \epsilon^{i-1}$ (locally on $R$) that are compatible with $\mathscr{L}$.

2. $\Rightarrow$ 3. Since $\mathscr{L}$ is projective of finite rank over $R[[\epsilon]]$, we find elements $g_1, \dots, g_r \in R[[\epsilon]]$ which generate the unit ideal and such that for all $i$, $\mathscr{L} \otimes_{R[[\epsilon]]} R[[\epsilon]]_{g_i}$ is free of rank $n$ over $R[[\epsilon]]_{g_i}$. Then the absolute terms $f_i := g_i(0) \in R$ generate the unit ideal of $R$, and for all $i$ with $f_i \neq 0$, we have that $g_i$ is a unit in $R_{f_i}[[\epsilon]]$, so

$$\mathscr{L} \otimes_{R[[\epsilon]]} R_{f_i}[[\epsilon]] = \mathscr{L} \otimes_{R[[\epsilon]]} R[[\epsilon]]_{g_i} \otimes_{R[[\epsilon]]_{g_i}} R_{f_i}[[\epsilon]]$$

is free of rank $n$ over $R_{f_i}[[\epsilon]]$.

3. $\Rightarrow$ 4. Trivial.

4. $\Rightarrow$ 1. Let $R \to R'$ be as in 4., and write $\mathscr{L}' = \mathscr{L} \otimes_{R[[\epsilon]]} R'[[\epsilon]]$. By assumption, $\mathscr{L}'$ is free over $R'[[\epsilon]]$, and in particular, for suitable $N$ we have

$t^N R'[[\epsilon]]^n \subseteq \mathscr{L}' \subseteq t^{-N} R'[[\epsilon]]^n$. By intersecting with $R[[\epsilon]]^n$, we obtain the analogous property for $\mathscr{L}$. Furthermore,

$$(\epsilon^{-N} R[[\epsilon]]/\mathscr{L}) \otimes_R R' = \epsilon^{-N} R'[[\epsilon]]/\mathscr{L}'$$

is locally free over $R'$, and since $R'$ is faithfully flat over $R$, $\epsilon^{-N} R[[\epsilon]]/\mathscr{L}$ is locally free over $R$. □

Example 2.9 shows that the affine Grassmannian for the multiplicative group $\mathbb{G}_m = \mathrm{GL}_1$ is not reduced. On the other hand, the affine Grassmannian for $\mathrm{SL}_n$ is integral (see [4], and [63], Corollary 5.3 and Theorem 6.1, which includes the case of positive characteristic and also deals with other groups).

Similarly as for the affine Grassmannian, we can now show that the affine flag variety $L\,\mathrm{GL}_n/\mathbf{I}$ is and ind-scheme. Here we let $A$ be the diagonal torus in $\mathrm{GL}_n$, we let $B$ be the Borel subgroup of upper triangular matrices, so that $B^-$ is the subgroup of lower triangular matrices. To describe $\mathcal{F}lag_{\mathrm{GL}_n}$ in terms of lattices, we use the notion of lattice chain:

**Definition 2.12.** Let $R$ be a $k$-algebra. A *(full periodic) lattice chain* inside $R((\epsilon))^n$ is a chain

$$\mathscr{L}_0 \supset \mathscr{L}_1 \supset \cdots \supset \mathscr{L}_{n-1} \supset \epsilon\mathscr{L}_0,$$

such that each $\mathscr{L}_i$ is a lattice in $R((\epsilon))^n$, and such that each quotient $\mathscr{L}_{i+1}/\mathscr{L}_i$ is a locally free $R$-module of rank 1.

Each element of $LG(R)$ gives rise to a lattice chain inside $R((\epsilon))^n$ by applying it to the standard lattice chain

$$\Lambda_{i,R} = \bigoplus_{j=0}^{n-i-1} R[[\epsilon]]e_{j+1} \oplus \bigoplus_{j=n-i}^{n-1} \epsilon R[[\epsilon]]e_{j+1},$$

where $e_1, \dots, e_n$ denotes the standard basis of $R((\epsilon))^n$. Because the Iwahori subgroup $\mathbf{I}(R) \subset \mathrm{GL}_n(R[[\epsilon]])$ is precisely the stabilizer of the standard lattice chain $\Lambda_{\bullet,R}$, we get:

**Proposition 2.13.** *Let $\mathcal{F}lag$ be the affine flag variety for $\mathrm{GL}_n$, and let $R$ be a $k$-algebra. Then $\mathcal{F}lag(R)$ is the space of lattice chains in $R((\epsilon))^n$. In particular, $\mathcal{F}lag$ is an ind-scheme. The affine flag variety for $\mathrm{SL}_n$ is the closed sub-ind-scheme of all lattice chains $(\mathscr{L}_\bullet)_\bullet$ such that $\mathscr{L}_0$ is 0-special.*

### 2.5. The Affine Grassmannian for an Arbitrary Linear Algebraic Group

To establish the existence of the affine Grassmannian as an ind-scheme in the general case, we will embed a given linear algebraic group into a general linear group in a suitable way. We use the following lemma due to Beilinson and Drinfeld.

**Lemma 2.14 ([7], Proof of Theorem 4.5.1).** *Let $G_1 \subset G_2$ be linear algebraic groups over $k$ such that the quotient $U := G_2/G_1$ is quasi-affine. Suppose that the quotient $LG_2/L^+G_2$ is an ind-scheme of ind-finite type. Then the same holds for $LG_1/L^+G_1$, and the natural morphism $LG_1/L^+G_1 \to LG_2/L^+G_2$ is a locally closed immersion. If $U$ is affine, then this immersion is a closed immersion.*

See also [63] Theorem 1.4 for a version which includes the twisted case. As an application of the lemma, we obtain:

**Proposition 2.15.** *Let $G$ be a linear algebraic group over $k$. Then the quotient $LG/L^+G$ is an ind-scheme over $k$. If $G$ is reductive, then it is ind-projective.*

*Proof.* First assume that $G$ is a reductive group, the case which will be relevant for us. Choose an embedding $G \to \mathrm{GL}_n$ of $G$ into some general linear group. Since $G$ is reductive, the quotient $\mathrm{GL}_n /G$ is affine. In fact, a quotient of a reductive group by a closed subgroup is affine if (and only if) the subgroup is reductive; see [70]. The lemma above, together with Proposition 2.10, then shows that $LG/L^+G$ is an ind-projective ind-scheme over $k$.

In the general case, one shows that there exists an embedding $G \to \mathrm{GL}_n \times \mathbb{G}_m$ such that the quotient is quasi-affine. See [7], Theorem 4.5.1 or [63] Proposition 1.3. □

**Remark 2.16.** 1. Note that in general, given a closed immersion of algebraic groups $G_1 \to G_2$, the induced morphism of the affine Grassmannians is not a closed immersion. For an example consider the inclusion of a Borel subgroup $B$ into a reductive group $G$. The morphism $LB/L^+B \to LG/L^+G$ is a bijection on $k$-valued points, but except for trivial cases is far from being an isomorphism.
2. For a different method of constructing the affine Grassmannian in all relevant cases, see Faltings' paper [21].

Similarly, we obtain that the affine flag variety $\mathcal{F}lag_G = LG/\mathbf{I}$ is an ind-scheme. We can either again use an embedding of $G$ into $\mathrm{GL}_n$ or $\mathrm{GL}_n \times \mathbb{G}_m$, or show that it is an ind-scheme by considering the natural projection $\mathcal{F}lag_G \to \mathcal{G}rass_G$ which is a fiber bundle whose fibers are all isomorphic to the usual flag variety of $G$.

Another interesting and important description of the affine Grassmannian for a semisimple group $G$ can be given in terms of $G$-bundles on a (smooth, projective) curve. See the surveys by Kumar [49] and Sorger [75] for introductions to this point of view. Here, we just sketch how the relationship is established. The basic idea is to glue vector bundles on the curve $C$ by gluing the trivial bundle on the pointed curve $C \setminus \{\mathrm{pt}\}$ and the trivial bundle on the formal neighborhood $\mathrm{Spec}\, k[[\epsilon]]$ of the point to obtain a vector bundle (or more generally, a $G$-bundle) on $C$. Note that this gluing is not an instance of faithfully flat descent, because in general the homomorphism $R \to R[[\epsilon]]$ is not flat. It is flat if $R$ is noetherian, or more generally if $R$ is a coherent ring (see [12] I §2, Ex. 12, [24]). For locally free modules, Beauville and Laszlo [5] have proved directly that descent holds in this situation. To prove that in this way one can construct all bundles, one uses the result of Drinfeld and Simpson who have shown that $G$-bundles on $C \setminus \{\mathrm{pt}\} \times S$ are trivial fpqc-locally on $S$. Here the assumption that $G$ is semisimple is obviously crucial (since for instance for $G = \mathbb{G}_m$, where $G$-bundles are just line bundles, this usually fails).

### 2.6. Decompositions

Below we need the following decompositions. Write $K = G(\mathcal{O})$, and denote by $X_{*,+}$ the subset of $X_*(A)$ consisting of all dominant coweights.

**Theorem 2.17 (Cartan decomposition).** *The affine Grassmannian decomposes as a disjoint union*

$$\mathcal{G}rass(k) = \bigcup_{\lambda \in X_{*,+}} K\epsilon^\mu K/K.$$

*The closure of each "Schubert cell" $K\epsilon^\mu K/K$ is a union of cells, and the closure relations are given by the partial order on dominant coweights introduced in Section* 1.1*:*

$$\overline{K\epsilon^\mu K/K} = \bigcup_{\lambda \leq \mu} K\epsilon^\lambda K/K$$

In the case of $G = \mathrm{GL}_n$, the translation element $\lambda$ such that $g \in K\epsilon^\lambda K$ is simply given by the elementary divisors of the lattice $g\Lambda_k$ with respect to the standard lattice $\Lambda_k$.

The corresponding result for the affine flag variety is the following. In fact, in this case the geometric structure of each cell is very simple. We denote by

$$I = \mathbf{I}(k) \subset G(\mathcal{O})$$

the Iwahori subgroup of $G(k((\epsilon)))$ given by $\mathbf{I}$.

**Theorem 2.18 (Iwahori-Bruhat decomposition).** *The affine flag variety decomposes as a disjoint union*

$$\mathcal{F}lag(k) = \bigcup_{x \in \widetilde{W}} IxI/I.$$

*The closure of each "Schubert cell" $IxI/I$ is a union of Schubert cells, and the closure relations are given by the Bruhat order:*

$$\overline{IxI/I} = \bigcup_{y \leq x} IyI/I.$$

*For every $x \in \widetilde{W}$, the cell $IxI/I$ is isomorphic to $\mathbb{A}^{\ell(x)}$.*

### 2.7. Connected Components

As we have seen in the example of $G = \mathrm{GL}_n$, the affine Grassmannian (and in fact, the loop group) of $G$ may have several connected components.

**Definition 2.19.** The *algebraic fundamental group* $\pi_1(G)$ of $G$ is the quotient of the cocharacter lattice $X_*(A)$ by the coroot lattice.

Similarly as for the loop group of a topological space, we have

**Proposition 2.20 ([7] Proposition 4.5.4; [63], Theorem 0.1).** *The set of connected components of the loop group can be identified with $\pi_1(G)$.*

We denote the map which maps a point to its connected component by $\kappa = \kappa_G\colon G(L) \to \pi_1(G)$. We can describe the map $\kappa$ explicitly as follows: given $g \in G(L)$, the Cartan decomposition as stated above says that there exists a unique $\lambda \in X_*(A)_+$ such that $g \in G(\mathcal{O})\epsilon^\lambda G(\mathcal{O})$. Then $\kappa(g)$ is the image of $\lambda$ under the natural projection $X_*(A) \to \pi_1(G)$. The reason is that the positive loop group is connected. For the same reason, the map $\kappa$ factors through the quotients $\mathcal{G}rass_G$ and $\mathcal{F}lag_G$, and we denote the resulting maps again by $\kappa$, or $\kappa_G$ if we want to indicate which group we refer to.

**Example 2.21.** If $G = \mathrm{GL}_n$, then we can identify $\pi_1(G)$ with the quotient of $\mathbb{Z}^n$ by the subgroup of elements $(x_1, \dots, x_n) \in \mathbb{Z}^n$ with $\sum x_i = 0$, and hence $\pi_1(G) \cong \mathbb{Z}$. The map $\kappa$ maps an element $g \in \mathrm{GL}_n(k((\epsilon)))$ to the valuation of its determinant.

The map $\kappa_G$ is sometimes called the "Kottwitz map" because of its appearance in Kottwitz' papers [44], [45] on isocrystals. Cf. also the section about the classification of $\sigma$-conjugacy classes in Section 4.2 below. Kottwitz works in the $p$-adic situation and therefore has to define this map in a different way, and also defines it for non-split groups; cf. [63] for a discussion of the relationship of the different definitions. Note that in [26], [27], the same map is denoted $\eta_G$.

### 2.8. The Bruhat-Tits Building

We give the definition of the Bruhat-Tits building for $G = \mathrm{PGL}_n$ (or $G = \mathrm{SL}_n$) which is sometimes useful to visualize subsets of the affine Grassmannian or of the affine flag variety. Bruhat and Tits developed this theory for arbitrary reductive groups over local fields. See the books by Garrett [22], and by Abramenko and Brown [1] (where the relevant buildings are called Euclidean and affine buildings, respectively).

We let $K$ denote a complete discretely valued field, and denote by $k$ its residue class field. For us, the relevant cases are either $K = L = k((\epsilon))$, where $k$ is algebraically closed, and $K = F = \mathbb{F}_q((\epsilon))$. But in contrast to the ind-scheme structure on the set $G(L)/G(\mathcal{O})$, the theory of the building works equally well over fields of mixed characteristic, say the field $\mathbb{Q}_p$ of $p$-adic numbers, or the completion of its maximal unramified extension $\widehat{\mathbb{Q}}_p^{\mathrm{ur}}$. We denote by $\mathcal{O}_K$ the valuation ring of $K$. As above, we have the notion of $\mathcal{O}_K$-lattice inside $K^n$.

**Definition 2.22.** The *Bruhat-Tits building for* $\mathrm{PGL}_n$ over $K$ is the simplicial complex $\mathcal{B}$ where

- The set $\mathcal{B}_0$ of vertices of $\mathcal{B}$ is the set of equivalence classes of $\mathcal{O}_K$-lattices $\mathscr{L} \subseteq K^n$, where the equivalence relation is given by homothety, i.e., $\mathscr{L} \sim \mathscr{L}'$ if and only if there exists $c \in K^\times$ such that $\mathscr{L}' = c\mathscr{L}$.
- A set $\{L_1, \dots, L_m\}$ of $m$ vertices is a simplex if there exist representatives $\mathscr{L}_i$ of $L_i$ such that

$$\mathscr{L}_1 \supset \mathscr{L}_2 \supset \cdots \supset \mathscr{L}_m \supset \epsilon\mathscr{L}_1.$$

The $n$-dimensional simplices are called alcoves. Clearly, every simplex is contained in an alcove. We say that a lattice $\mathscr{L} \subset K^n$ is adapted to a basis $f_1, \dots, f_n$

of $K^n$, if $\mathscr{L}$ has an $\mathcal{O}_K$-basis of the form $\epsilon^{i_1} f_1, \dots, \epsilon^{i_n} f_n$. The apartment corresponding to the basis $f_i$ is the subcomplex of $\mathcal{B}$ whose simplices consist of vertices given by lattices adapted to this basis. The elementary divisor theorem shows that given any two vertices of $\mathcal{B}$, there exists an apartment containing both. In fact, one can show that given any two simplices, there exists an apartment containing both. The apartment corresponding to the standard basis is called the standard apartment.

The standard apartment is closely connected with the affine root system of $G$. The geometric realization of the standard apartment can be naturally identified with $X_*(A)_{\mathbb{R}}$. The set of vertices lying in the standard apartment can be identified with $X_*(A) = \mathbb{Z}^n/\mathbb{Z}\cdot(1,\dots 1)$, the residue class of $(i_1, \dots, i_n)$ corresponding to the homothety class of $\bigoplus_\nu \epsilon^{i_\nu} \mathcal{O}_K e_\nu$. The grid of affine root hyperplanes gives rise to the simplicial structure; see the figures on pages 40, 41, and 42 for the irreducible root systems of rank 2. In fact, the set of alcoves is equal to the set of connected components of the complement of the union of all affine root hyperplanes

$$\{x \in X_*(A)_{\mathbb{R}};\ \langle x, \alpha\rangle = i\}, \quad \alpha \in \Phi,\ i \in \mathbb{Z}.$$

Our choice of a standard lattice chain, or equivalently the choice of an Iwahori subgroup, gives us a distinguished alcove, called the *base alcove*. The extended affine Weyl group acts on the set of alcoves in the standard apartment. The affine Weyl group acts simply transitively, and hence can be identified with the set of alcoves, using the base alcove as a base point. On the other hand, the elements of length 0 are exactly those elements which fix the base alcove. (For instance, in the case of $\mathrm{PGL}_3$, there are 2 non-trivial rotations with center the barycenter of the base alcove, fixing the base alcove.) The length of an element $x$ of the (extended) affine Weyl group is the number of affine root hyperplanes separating the base alcove from the image of the base alcove under $x$. For a vertex of $\mathcal{B}$ represented by $\mathscr{L} = g\Lambda$, $g \in \mathrm{PGL}_n(K)$, we call the residue class of $\mathrm{val}(\det g)$ in $\mathbb{Z}/n$ the type of the vertex. This number is independent of the choice of representative.

For $n = 2$, the situation is particularly simple:

**Example 2.23.** Let $G = \mathrm{PGL}_2$. In this case, the simplicial complex $\mathcal{B}$ is a tree, as is easily seen using the notion of distance below. The notion of type gives $\mathcal{B}$ the structure of a bipartite graph: all neighbors of a vertex of type 0 have type 1, and conversely. The set of neighbors of a point represented by $\mathscr{L}$ can obviously be identified with the projective line $\mathbb{P}^1(k)$. We have an obvious notion of *distance* between two vertices: Given vertices represented by $\mathscr{L} \subseteq \mathscr{L}'$, we can choose bases of the form $b_1, b_2$, and $\epsilon^{d_1} b_1, \epsilon^{d_2} b_2$ of $\mathscr{L}$ and $\mathscr{L}'$, and the distance is given by $|d_1 - d_2|$ (which is independent of the choice of bases). See Figure 1 for an illustration.

The action of $\mathrm{PGL}_n(K)$, or $\mathrm{SL}_n(K)$, on $\mathcal{B}$ induces identifications

- $\mathcal{F}lag_{\mathrm{SL}_n}(k) =$ set of all alcoves in $\mathcal{B}$,
- $\mathcal{G}rass_{\mathrm{PGL}_n}(k) =$ set of all vertices in $\mathcal{B}$,
- $\mathcal{G}rass_{\mathrm{SL}_n}(k) =$ set of all vertices of type 0 in $\mathcal{B}$.

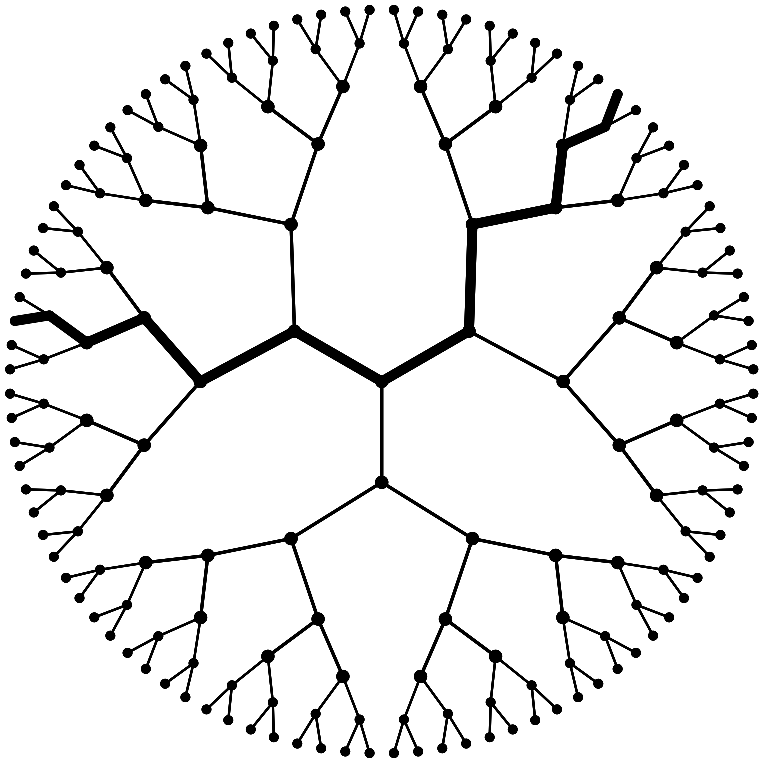

FIGURE 1. Part of the Bruhat-Tits tree for $\mathrm{PGL}_2$ over $\mathbb{F}_2((\epsilon))$ (or over $\mathbb{Q}_2$). One apartment is marked by thick lines.

## 3. Affine Springer Fibers

### 3.1. Springer Fibers

We start with a brief review of the classical theory of Springer fibers which provides important motivation for the notion of affine Springer fiber. A survey of this topic was given by Springer [74].

Let $k$ be an algebraically closed field of characteristic 0, and let $G$ be a connected semisimple algebraic group over $k$. One can also work in positive characteristic, provided one makes an assumption that the characteristic is sufficiently large with respect to the group.

We fix a maximal torus $A$ of $G$ and a Borel subgroup $B \subset G$ which contains $A$. We denote the Lie algebras of these groups by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{a}$, respectively. We have $\mathfrak{b} = \mathfrak{a} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the Lie algebra of the unipotent radical of $B$.

Denote by $\tilde{\mathfrak{g}}$ the quotient $G \times^B \mathfrak{b}$ of $G \times \mathfrak{b}$ by the $B$-action given by $b.(g,x) = (gb^{-1}, \mathrm{Ad}(b)x)$.

**Theorem 3.1 (Grothendieck, see** [74] **Theorem 1.4).** *The diagram*

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\varphi} & \mathfrak{g} \\ \downarrow{\scriptstyle\vartheta} & & \downarrow{\scriptstyle\chi} \\ \mathfrak{a} & \xrightarrow{\psi} & \mathfrak{a}/W \end{array}$$

*where*

- $\varphi$ *maps* $(g, x)$ *to* $\mathrm{Ad}(g)(x)$*,*
- $\vartheta$ *maps* $(g, x)$ *to the* $\mathfrak{a}$*-component of* $x \in \mathfrak{b} = \mathfrak{a} \oplus \mathfrak{n}$*,*
- $\chi$ *is the map induced by the homomorphism* $k[\mathfrak{a}/W] = k[\mathfrak{a}]^W \cong k[\mathfrak{g}]^G \subset k[\mathfrak{g}]$.
- $\psi$ *is the canonical projection from* $\mathfrak{a}$ *to the quotient of* $\mathfrak{a}$ *by* $W$*,*

*is a* simultaneous resolution *of* $\chi$*, i.e.,*

1. $\varphi$ *is proper,* $\vartheta$ *is smooth,* $\psi$ *is finite, and*
2. *for all* $a \in \mathfrak{a}$*, the morphism* $\vartheta^{-1} \to \chi^{-1}(\psi(a))$ *is a resolution of singularities (and in particular induces an isomorphism over the smooth locus of* $\chi^{-1}(\psi(a))$*).*

If $G = \mathrm{GL}_n$, then we can identify $\mathfrak{a}/W$ with affine space $\mathbb{A}^n$, and $\chi$ with the map which sends $x \in \mathfrak{gl}_n = \mathrm{Mat}_{n\times n}(k)$ to the coefficients of its characteristic polynomial. Then $\chi^{-1}(0)$ is the nilpotent cone, the subset of all nilpotent matrices.

**Definition 3.2.** The fibers of $\varphi$ are called *Springer fibers*. For $x \in \mathfrak{g}$, we write $\varphi^{-1}(x) = Y_x$ (considered as a reduced scheme).

Note that for $x \in \mathfrak{g}$, we can rewrite the Springer fiber $Y_x$ as

$$Y_x = \{g \in G/B;\ \mathrm{Ad}(g^{-1})(x) \in \mathfrak{b}\}, \quad (g, z) \mapsto g, \tag{1}$$

the inverse map being given by $g \mapsto (g, \mathrm{Ad}(g^{-1})(x))$. We note some properties of Springer fibers:

**Proposition 3.3 (see** [74]**, Theorem 1.8).** *Let* $x \in \mathfrak{g}$*, and denote by* $H = Z_G(x_s)^0$ *the identity component of the centralizer of the semisimple part* $x_s$ *of* $x$*.*

1. *The number of connected components of* $Y_x$ *is* $\#W/W_x$*, where* $W_x$ *is the Weyl group of* $H$*.*
2. *Each connected component of* $Y_x$ *is isomorphic to* $Y^H_{x_n}$*, where* $x_n$ *is the nilpotent part of* $x$*.*
3. $Y_x$ *is equidimensional of dimension* $\frac{1}{2}(\dim Z_G(x) - \mathrm{rk}\, G)$*.*

Part 3., which is the hardest, is due to Spaltenstein. It implies in particular that $\varphi$ is a "small morphism". An interesting application of Springer fibers is the construction of the irreducible representations of the Weyl group of $G$.

Springer fibers are usually very singular varieties. All the more surprising is the following purity theorem (see Section 3.5 for a brief discussion of purity):

**Theorem 3.4 (Spaltenstein [72]).** *Let $G = \mathrm{GL}_n$. Then every Springer fiber $Y_x$ admits a paving by affine spaces. In particular, its ($\ell$-adic) cohomology is concentrated in even degrees and is pure.*

If one works over the field of complex numbers, one can replace $\ell$-adic cohomology by singular cohomology. Shimomura [71] has generalized the theorem to Springer fibers for $\mathrm{GL}_n$ in partial flag varieties, see also the paper [39] by Hotta and Shimomura. On the other hand, the Springer fibers have severe singularities, and in particular Poincaré duality fails for these varieties, even on the level of Betti numbers.

### 3.2. Affine Springer Fibers

Now let $k$ be an algebraically closed field, let $\mathcal{O} = k[[\epsilon]]$, and let $L = k((\epsilon))$ be the field of Laurent series. (The hypothesis that $k$ is algebraically closed is not necessary, and not even desirable for some applications; we make it here to simplify the situation a little bit.) We fix a connected reductive linear algebraic group $G$ over $k$, and a maximal torus $A \subseteq G$. We denote by $\mathfrak{g}$ and $\mathfrak{a}$ the Lie algebras of $G$ and $A$, resp. There is no widely accepted notation for affine Springer fibers. Often they are denoted by $\mathcal{G}rass_\gamma$ (which is of course more appealing when the affine Grassmannian is denoted by a shorter symbol like $X$ (as in [41]) or $\mathfrak{X}^G$ (as in [15]). We will denote affine Springer fibers by $\mathcal{F}(\gamma)$, a notation which is close to the notation of [32], and make the following definition (cf. 1):

**Definition 3.5.** The *affine Springer fiber* associated with $\gamma \in \mathfrak{g}(L)$ is

$$\mathcal{F}(\gamma) = \{x \in G(L);\ \mathrm{Ad}(x^{-1})\gamma \in \mathfrak{g}(\mathcal{O})\}/G(\mathcal{O}),$$

a locally closed subset of $\mathcal{G}rass(k)$. We view $\mathcal{F}(\gamma)$ as an ind-scheme over $k$ by giving it the reduced ind-scheme structure.

**Remark 3.6.**
1. Note that all of the following cases can occur, depending on $\gamma$: $\mathcal{F}(\gamma) = \emptyset$, $\mathcal{F}(\gamma)$ is a scheme of finite type over $k$, $\mathcal{F}(\gamma)$ is a scheme locally of finite type (but not of finite type) over $k$, $\mathcal{F}(\gamma)$ is not a scheme (but only an ind-scheme).
2. In [61], Ngô gives a functorial definition of affine Springer fibers in terms of $G$-bundles, and hence obtains a natural ind-scheme structure.
3. There is a variant of the definition where the Lie algebra $\mathfrak{g}(\mathcal{O})$ of the maximal compact subgroup $G(\mathcal{O})$ is replaced by the Lie algebra of an Iwahori subgroup. Then we obtain affine Springer fibers in the affine flag variety of $G$.
4. Another obvious variant is to replace the Lie algebra with the group itself, and to replace the adjoint action by the conjugation action. At least if the characteristic is 0 or sufficiently large, then one can switch back and forth between these two points of view using a quasi-logarithm, see e. g. the work of Kazhdan and Varshavsky, [42], 1.8.

Similarly as in Proposition 3.3 2., one can often reduce to the case that $\gamma$ is a *topologically nilpotent* element, i.e., that $\gamma^n$ converges to 0 in the $\epsilon$-adic topology, using the topological Jordan decomposition. See Spice's paper [73] for details in the group (rather than the Lie algebra) case.

### 3.3. General Properties

First note that multiplication by $g$ induces an isomorphism $\mathcal{F}(\gamma) \cong \mathcal{F}(\mathrm{Ad}(g)\gamma)$, so we can study all non-empty affine Springer fibers, up to isomorphism, by considering $\gamma \in \mathfrak{g}(O)$.

Recall that a semisimple element in $\mathfrak{g}(L)$ is called regular, if its centralizer is a maximal torus (i.e., if the centralizer is "as small as possible"). In the case of $\mathrm{GL}_n$ this just means that all eigenvalues over an algebraic closure are different. Although $G$ is split, the centralizer of a regular semisimple element $\gamma$ will not be a *split* maximal torus in general, because the field $k((\epsilon))$ is not algebraically closed. If $\operatorname{char} k = 0$, then the algebraic closure of $k((\epsilon))$ is the field of Puiseux series,

$$\overline{k((\epsilon))} = \bigcup_{e \in \mathbb{Z}_{\geq 0}} k((\epsilon^{\frac{1}{e}})).$$

If $\operatorname{char} k > 0$, then the field on the right hand side is the perfect closure of the maximal tamely ramified extension of $k((\epsilon))$. See Kedlaya's paper [43] for a description of the algebraic closure and for further references.

For the remainder of Section 3, we make the assumption that the order $\#W$ of the Weyl group of $G$ is invertible in $k$. This implies that, even if $\operatorname{char} k > 0$, every maximal torus of $G$ splits over a tamely ramified extension of $L$, i.e., over an extension of the form $k((\epsilon^{\frac{1}{e}}))$, where $\operatorname{char} k$ does not divide $e$. See [32] for a more thorough discussion of the situation in positive characteristic.

The following proposition shows that only affine Springer fibers for $\gamma$ regular semisimple are "reasonable" geometric objects, at least for our purposes:

**Proposition 3.7 ([41], §2 Corollary, Lemma 6).** *Let $\gamma \in \mathfrak{g}(\mathcal{O})$. We have*

$$\dim \mathcal{F}(\gamma) < \infty \iff \gamma \in \mathfrak{g}(L) \text{ regular semisimple.}$$

From now on, $\gamma$ will always be assumed to be regular semisimple. In this case, we get more precise information:

**Theorem 3.8 ([41]; [9]).** *Let $\gamma \in \mathfrak{g}(\mathcal{O})$ be regular semisimple (as an element of $\mathfrak{g}(L)$).*

1. *Let $T$ be the centralizer of $\gamma$ in $G(k((\epsilon)))$, a maximal torus. Let $A_\gamma$ be the maximal split subtorus of $T$. Then $\mathrm{Hom}_{k((\epsilon))}(\mathbb{G}_m, T) = X_*(T) = X_*(A_\gamma)$ acts freely on $\mathcal{F}(\gamma)$, and the quotient $X_*(A_\gamma)\backslash\mathcal{F}(\gamma)$ is a projective $k$-scheme.*
2. *In particular, if $\gamma$ is elliptic, i.e., $X_*(A_\gamma) = 0$, then $\mathcal{F}(\gamma)$ is a projective $k$-scheme.*
3. *Let $\mathfrak{z}(\gamma) \subseteq \mathfrak{g}(L)$ be the centralizer of $\gamma$, and let $\delta_\gamma$ be the map*

$$\delta_\gamma = \mathrm{ad}(\gamma)\colon \mathfrak{g}(L)/\mathfrak{z}(\gamma) \to \mathfrak{g}(L)/\mathfrak{z}(\gamma).$$

*Then*

$$\dim \mathcal{F}(\gamma) = \frac{1}{2}\left(\mathrm{val}(\det(\delta_\gamma)) - \mathrm{rk}\,\mathfrak{g} + \dim \mathfrak{a}^w\right),$$

*where $w$ denotes the type of $\mathfrak{z}(N)$ (see* [41], *§1, Lemma 2 or* [32] *5.2) and $\mathfrak{a}^w$ is the fix point locus of $w$ in the Lie algebra $\mathfrak{a}$ of the fixed maximal torus $A$.*

A different way of obtaining the dimension formula is given by Ngô, [61] 3.8. It was proved by Kazhdan and Lusztig ([41], §4 Proposition 1) that affine Springer fibers in the Iwahori case are equidimensional. Equidimensionality in the Grassmannian case was proved by Ngô, see [61] Proposition 3.10.1.

For a moment, let us consider affine Springer fibers over a finite base field. One reason why this is interesting is that the number of points of a quotient $X_*(A_\gamma)\backslash\mathcal{F}(\gamma)$ over a finite field can be expressed as an orbital integral:

$$\#(X_*(A_\gamma)\backslash\mathcal{F}(\gamma))(\mathbb{F}_q) = \#X_*(A_\gamma)\backslash\{g \in G(F);\ \mathrm{Ad}(g^{-1})\gamma \in \mathfrak{g}(\mathfrak{o})\}/G(\mathcal{O})$$
$$= \int_{T(F)\backslash G(F)} \mathbf{1}_{\mathfrak{g}(\mathfrak{o})}(\mathrm{Ad}(g^{-1})\gamma)\, dg/dt =: O_\gamma(\mathbf{1}_{\mathfrak{g}(\mathfrak{o})})$$

for measures $dg$, $dt$ such that $G(\mathcal{O})$ and $T(\mathcal{O})$ have volume 1. Such orbital integrals are of great interest from the point of view of the Langlands program, and more specifically of the fundamental lemma, a long-standing conjecture of Langlands and Shelstad which was recently proved by Ngô, [61]. Ngô does use affine Springer fibers along the way of his proof, but as Example 3.4.2 by Bernstein and Kazhdan suggests, it is hopeless to "compute" the number of points of an affine Springer fiber directly.

By the Grothendieck-Lefschetz fix point formula, one can express the number of points of a variety over a finite field in terms of the trace of the Frobenius morphism on the $\ell$-adic cohomology. We will report on several results on the cohomology of affine Springer fibers in Sections 3.5–3.7.

For classical groups, one can also express these numbers in a completely elementary way, as numbers of lattices satisfying certain conditions.

### 3.4. Examples

**3.4.1.** $\mathrm{SL}_2$**.** For more details on the example in this section see [31] 6. We fix an algebraically closed base field $k$ of characteristic $\neq 2$, let $G = \mathrm{SL}_2$, denote by $A$ the diagonal torus, by $\mathcal{G}rass$ the corresponding affine Grassmannian, and by $x_0 \in \mathcal{G}rass$ the base point corresponding to the standard lattice $\Lambda = k[[\epsilon]]^2$.

Denote by $\alpha$ the unique positive root (with respect to the Borel subgroup of upper triangular matrices). We regard $\alpha$ as the morphism $A \to \mathbb{G}_m$, $\mathrm{diag}(a, a^{-1}) \mapsto a^2$. We denote by $\alpha'$ its "differential", i.e., the homomorphism

$$\alpha' \colon \mathfrak{a}(L) \to L, \qquad \begin{pmatrix} a & \\ & -a \end{pmatrix} \mapsto 2a$$

For $n \leq -1$, we set $x_n = \begin{pmatrix} 1 & \varepsilon^n \\ & 1 \end{pmatrix}$.

**Lemma 3.9 (Nadler, [31] Lemma 6.2).** 1. *The affine Grassmannian is the disjoint union*

$$\mathcal{G}rass = \bigcup_{n \le 0} A(L)x_n,$$

*and for each $n \le 0$, we have* $\dim A(L)x_n = |n|$.

2. *For $\gamma \in \mathfrak{a}(\mathcal{O})$, setting $v = \mathrm{val}(\alpha'(\gamma))$, we have*

$$\mathcal{F}(\gamma) = \bigcup_{n=-v}^{0} A(L)x_n.$$

*Proof.* The proof is elementary; see [31]. In terms of the Bruhat-Tits tree for $\mathrm{SL}_2$, we can understand the lemma as follows: The action of $A(L)$ on the tree fixes the standard apartment, and hence preserves the distance to the standard apartment. One checks that $x_n$ has distance $|n|$ to the standard apartment, and to prove the first part, one has to prove that $A(L)$ acts transitively on the set of points of a fixed distance to the standard apartment.

It is clear that $A(L)$ acts on $\mathcal{F}(\gamma)$, so $\mathcal{F}(\gamma)$ is a union of $A$-orbits, and the lemma says, from the point of view of the building, that $\mathcal{F}(\gamma)$ is the set of all points of type 0 of distance $\le \mathrm{val}(\alpha'(\gamma))$ from the standard apartment. □

We see that for $\mathrm{val}(\alpha'(\gamma)) = 0$, $\mathcal{F}(\gamma) \cong X_*(A)$ is a discrete set, while for $\mathrm{val}(\alpha'(\gamma)) = 1$ it is a chain of countably many projective lines where the $i$-th chain and the $(i+1)$-th chain intersect transversally in a single point, and no other intersections occur. We can take the quotient $\mathbb{Z}\backslash\mathcal{F}(\gamma)$ and obtain a nodal rational curve. This illustrates Theorem 3.8, 1.

Of course, one checks immediately that the above is consistent with the general dimension formula stated above: for $\gamma \in \mathfrak{a}(\mathcal{O})$ regular, with the notation of Theorem 3.8, $\mathfrak{z}(\gamma) = \mathfrak{a}$, and $w = \mathrm{id}$. The factor $\frac{1}{2}$ arises because $\delta_\gamma$ is defined on the whole (positive and negative) root space.

One can now go on to describe the $A(k)$-fix points and orbits in $\mathcal{F}(\gamma)$, and hence compute its $T$-equivariant homology; see [31] 7 and Section 3.7 below. Since there is only one positive root, for $\mathrm{SL}_2$ one is always in the "equivaluation" case, see below.

**3.4.2. The Example of Bernstein and Kazhdan.** In the appendix to [41], J. Bernstein and D. Kazhdan give an example of an affine Springer fiber $\mathcal{F}(\gamma)$ in the affine flag variety of $G = \mathrm{Sp}_6$ which is not a rational variety. More precisely, it has an irreducible component which admits a dominant morphism to an elliptic curve (whose isomorphism class depends on the element $\gamma$). This shows that one cannot expect to have a closed formula for the number of points of an affine Springer fiber over a finite field. Furthermore, this affine Springer fiber cannot have a paving by affine spaces.

### 3.5. Purity

In this section, $k$ denotes an algebraic closure of $\mathbb{F}_p$. Whenever we consider $\ell$-adic cohomology, we assume that $\ell$ is a prime different from $p$.

Let $X_0$ be a separated $\mathbb{F}_q$-scheme of finite type, and let $X = X_0 \otimes_{\mathbb{F}_q} k$. The geometric Frobenius $\mathrm{Fr} \in \mathrm{Gal}(k/\mathbb{F}_q)$, i.e., the inverse of the usual ("arithmetic") Frobenius morphism $x \mapsto x^q$, acts on $X$ via its action on the second factor of the product $X_0 \otimes_{\mathbb{F}_q} k$, and hence on the $\ell$-adic cohomology groups $H^i(X) := H^i(X, \overline{\mathbb{Q}}_\ell)$. The cohomology is called pure, if for every integer $i$, the space $H^i(X, \overline{\mathbb{Q}}_\ell)$ is pure of weight $n$ in the sense of Deligne: For every embedding $\iota\colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and every eigenvalue $\alpha$ of Fr on $H^i(X)$, $|\iota(\alpha)| = q^{i/2}$. Note that this is really a property of $X$; it is independent of the choice of $X_0$ and $q$. Every $X$ of finite type over $k$ is defined over some finite field.

There is a large class of varieties with pure cohomology:

**Theorem 3.10 (Deligne).** *Let $X$ be a smooth and proper $k$-scheme. Then the cohomology of $X$ is pure.*

On the other hand, the cohomology of a singular variety will usually not be pure. Springer fibers and affine Springer fibers are (expected to be) exceptions to this rule. Another standard method of checking purity is to show that the variety in question admits a paving by affine spaces. Since affine Springer fibers can have cohomology in odd degrees, they cannot have such a paving in general, however; cf. also Example 3.4.2. See Section 3.7 for positive results.

### 3.6. (Co-)homology of Ind-Schemes

In the sequel, it will sometimes be more useful to use homology rather than cohomology. One defines

$$H_i(X) = H_i(X, \overline{\mathbb{Q}}_\ell) = \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell}(H^i(X), \overline{\mathbb{Q}}_\ell) = H_c^{-i}(X, K_X),$$

where $K_X$ denotes the dualizing complex of $X$, and the final equality is given by Poincaré duality. Cf. [15] 3.3.

If $X$ is an ind-scheme, say $X = \bigcup_n X_n$, with $X_n$ of finite type and separated, then we set

$$H^i(X) = \varprojlim H^i(X_n),$$

and

$$H_i(X) = \varinjlim H_i(X_n).$$

These groups are independent of the choice of representation of $X$ as a union of finite-dimensional schemes.

### 3.7. Equivariant Cohomology

One of the important tools in studying cohomological properties of affine Springer fibers is equivariant cohomology. The centralizer $T$ of $\gamma$ acts on the affine Springer fiber $\mathcal{F}(\gamma)$, and equivariant cohomology takes into account the additional structure given by this action. Under a purity assumption, the equivariant cohomology is completely encoded by the 0- and 1-dimensional orbits of $T$; see the Lemma of

Chang and Skjelbred (Proposition 3.15). Furthermore, in favorable situations, for instance if the cohomology is pure, the usual cohomology can easily be recovered from the equivariant one. We sketch the definition of equivariant cohomology in the $\ell$-adic setting. Though elegant, it is not easy to digest because it uses $\ell$-adic cohomology of algebraic stacks. As long as one works over the field of complex numbers, one can also use the classical topological version of equivariant cohomology, see [30] and Tymoczko's introductory paper [76]. The reference we follow in the $\ell$-adic setting is the paper [15] by Chaudouard and Laumon.

Let $k$ be an algebraic closure of the field $\mathbb{F}_q$ with $q$ elements, let $p = \operatorname{char} k$. Let $X$ be a separated $k$-scheme of finite type, and let $T$ be an algebraic torus acting on $X$.

**Definition 3.11.** The *$T$-equivariant $\ell$-adic cohomology groups* of $X$ are

$$H^n_T(X) := H^n_T(X, \overline{\mathbb{Q}}_\ell) := H^n([X/T], \overline{\mathbb{Q}}_\ell),$$

where $[X/T]$ denotes the stack quotient of $X$ by the action of $T$.

For $X = \operatorname{Spec} k$, the Chern-Weil isomorphism describes the equivariant cohomology (with respect to the trivial action by $T$). This is particularly important, because for any $X$, writing $H^*_T(X) := \bigoplus_{n\geq 0} H^n_T(X)$, cup-product induces on $H^*_T(X)$ the structure of a graded algebra, and of a $H^*_T(\operatorname{Spec} k)$-module. To state the Chern-Weil isomorphism, we define

$$\mathscr{D}^* := \operatorname{Sym}^*(X^*(T) \otimes \overline{\mathbb{Q}}_\ell(-1)).$$

Here $\overline{\mathbb{Q}}_\ell(-1)$ is the Tate twist, i.e., the vector space $\overline{\mathbb{Q}}_\ell$ where the geometric Frobenius acts by multiplication by $q$.

**Theorem 3.12 (Chern-Weil isomorphism).** *There is a natural isomorphism*

$$\mathscr{D}^* \longrightarrow H^*_T(\operatorname{Spec} k)$$

*doubling the degree, i.e.,*

$$H^i_T(\operatorname{Spec} k) = \begin{cases} \operatorname{Sym}^j(X^*(T) \otimes \overline{\mathbb{Q}}_\ell(-1)) & \text{if } i = 2j \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

*Proof.* We give a sketch of the proof; see Behrend's paper [8], 2.3, for details. Using the Künneth formula, one reduces to the case $T = \mathbb{G}_m$. Now $[\mathbb{A}^{N+1}_k/\mathbb{G}_m]$ has the same cohomology as $[\operatorname{Spec} k/\mathbb{G}_m]$, and the open immersion

$$\mathbb{P}^N = (\mathbb{A}^{N+1} \setminus \{0\})/\mathbb{G}_m \to [\mathbb{A}^{N+1}/\mathbb{G}_m]$$

gives us, by purity, that

$$H^i([\mathbb{A}^{N+1}/\mathbb{G}_m]) \cong H^i(\mathbb{P}^N) \cong \overline{\mathbb{Q}}_\ell(-i)$$

for all $i \leq 2N$. □

Now we return to the case of an arbitrary separated $k$-scheme $X$ of finite type on which $T$ acts. The Leray spectral sequence for the natural morphism $[X/T] \to [\operatorname{Spec} k/T]$ has the form

$$E_2^{p,q} = H_T^p(\operatorname{Spec} k) \otimes H^q(X) \Longrightarrow H_T^{p+q}(X). \tag{2}$$

**Definition 3.13.** The scheme $X$ (together with the given $T$-action) is called *equivariantly formal*, if the spectral sequence (2) degenerates at the $E_2$ term.

If $X$ is equivariantly formal, then

$$H_T^*(X) \cong H^*(X) \otimes H_T^*(\operatorname{Spec} k),$$
$$H^*(X) \cong H_T^*(X) \otimes_{H_T^*(\operatorname{Spec} k)} \overline{\mathbb{Q}}_\ell,$$

so the usual and the equivariant cohomology determine each other in a simple way. Because the differentials in the spectral sequence respect the Frobenius action, we obtain

**Proposition 3.14.** *Let $X$ be as above, and assume that $H^*(X) = H^*(X, \overline{\mathbb{Q}}_\ell)$ is pure. Then $X$ is equivariantly formal.*

Similarly as for usual cohomology, we define

$$H_i^T(X) = \operatorname{Hom}_{\overline{\mathbb{Q}}_\ell}(H_T^n(X), \overline{\mathbb{Q}}_\ell),$$

and if $X = \bigcup_n X_n$ is an ind-scheme with $X_n$ separated, of finite type, we define

$$H_T^i(X) = \varprojlim H_T^i(X_n), \qquad H_i^T(X) = \varinjlim H_i^T(X_n),$$

Equivariant cohomology incorporates the additional structure given by the torus action of $T$ on $X$. One way to make use of this is the following:

**Proposition 3.15 (Lemma of Chang-Skjelbred,** [15] **Lemme 3.1).** *Let $V$ be a finite-dimensional vector space on which $T$ acts algebraically, and let $X \subseteq \mathbb{P}(V)$ be a $T$-stable closed subscheme of the projective space of lines in $V$. Suppose that the cohomology of $X$ is pure. Denote by $X_0$ the set of $T$-fix points in $X$, and by $X_1$ the union of all orbits of $T$ of dimension $\leq 1$.*

*There is an exact sequence*

$$H_\bullet^T(X_1, X_0) \to H_\bullet^T(X_0) \to H_\bullet^T(X) \to 0,$$

*where $H_\bullet^T(X_1, X_0)$ denotes the relative equivariant homology (see* [15] *3.5).*

Therefore for pure varieties with torus action we can compute the equivariant homology, and hence the homology, once we understand the 1-dimensional and 0-dimensional $T$-orbits. This makes the following purity conjecture a central topic in the theory of affine Springer fibers.

**Conjecture 3.16 (Goresky, Kottwitz, MacPherson** [31] **Conj. 5.3).** *For every $n \geq 0$, the homology group $H_n(\mathcal{F}(\gamma))$ is pure of weight $-n$.*

Assuming the conjecture, the usual cohomology is related to the equivariant cohomology as explained above, and since the equivariant cohomology can be described in terms of the fixed points and one-dimensional orbits of the torus, a detailed study of the torus action yields an explicit description of the cohomology of affine Springer fibers. In the case where $\gamma$ is unramified (i.e., its centralizer is split over $L$, and hence can be assumed to be equal to $A$), Goresky, Kottwitz and MacPherson have proved:

**Theorem 3.17** ([31] **Theorem 9.2).** *Let $\gamma \in \mathfrak{a}(\mathcal{O})$, and assume that $\mathcal{F}(\gamma)$ is pure. Then Proposition* 3.15 *induces an isomorphism*

$$H_\bullet^A(\mathcal{F}(\gamma)) \cong (k[X_*(A)] \otimes \mathscr{D}^*)/\sum_{\alpha\in\Phi^+} L_{\alpha,\gamma},$$

*where*

$$L_{\alpha,\gamma} = \sum_{d=1}^{\mathrm{val}(\alpha'(\gamma))} (1-\alpha^\vee)k[X_*(A)] \otimes \mathscr{D}^*\{\partial_\alpha^d\}.$$

Here $\partial_\alpha$ is the differential operator of degree 1 on $\mathscr{D}^*$ corresponding to $\alpha$. Note that the set of $A$-fixed points in $\mathcal{F}(\gamma)$ is equal to the set of all $A$-fixed points in $\mathcal{G}rass$, and hence can be identified with $X_*(A)$. Therefore its homology is just $k[X_*(A)] \otimes \mathscr{D}^*$. See loc. cit. for details.

In the "equivalued" case, Goresky, Kottwitz and MacPherson have proved the purity conjecture. Let $\gamma \in \mathfrak{g}(L)$ be a regular semisimple element with centralizer $T$. The element $\gamma$ is called *integral*, if $\mathrm{val}(\lambda'(\gamma)) \geq 0$ for every $\lambda \in X^*(T)$, and it is called *equivalued*, if for every root $\alpha$ of $T$ (over an algebraic closure of $L$), the valuation $\mathrm{val}(\alpha'(\gamma))$ is equal to some constant $s$ independent of $\alpha$, and $\mathrm{val}(\lambda'(\gamma)) \geq s$ for every $\lambda \in X^*(T)$.

**Theorem 3.18 (**[32]**, Theorem 1.1).** *Assume that $p$ does not divide the order of $W$. Let $\gamma$ be an integral equivalued regular element of* $(\mathrm{Lie}\, T)(L)$*, where $T \subset G_L$ is a maximal torus. Then the affine Springer fiber $\mathcal{F}(\gamma)$ has pure cohomology.*

The theorem is proved by showing that the affine Springer fibers in question admit a paving by varieties which are fiber bundles in affine spaces over certain smooth, projective varieties, and invoking Deligne's Theorem 3.10. V. Lucarelli [56] has provided examples of affine Springer fibers for $\mathrm{PGL}_3$ and elements of unequal valuation which admit pavings by affine spaces, thus proving the purity conjecture in these cases.

### 3.8. The Fundamental Lemma

The fundamental lemma, sometimes called, more appropriately, the matching conjecture of Langlands and Shelstad, is a family of combinatorial identities which relate orbital integrals (of different sorts) for different groups, which was conjectured by Langlands and Shelstad [50]. The precise statement is quite complicated; see loc. cit. and Hales' paper [35]. The relationship between the groups occurring is given by "endoscopy". Endoscopic groups for $G$ are described in terms of the root datum of $G$, and there is no simple relationship in terms of the groups themselves. As an example, if $G = U(n)$ is a quasi-split unitary group, then the products $U(n_1) \times U(n_2)$, $n_1 + n_2 = n$, are endoscopic groups for $n$. Note however that in general an endoscopic group is not a subgroup of the given group.

Before the work of Ngô, see below, the fundamental lemma had been proved in several special cases; see [35] for references. Furthermore, results of Langlands-Shelstad, Hales and Waldspurger allowed to reduce the originally $p$-adic statement to a Lie algebra version in the function field case over fields of high positive characteristic. The idea of translating the fundamental lemma into an algebro-geometric statement and using the highly developed machinery of algebraic geometry to prove it has been around for some time. For instance, see the paper [54] by Laumon and Rapoport. It was not clear until quite recently, though, how to translate the complicated combinatorics to "simple" geometry, rather than to intractable geometry.

We have seen above that the number of points of an affine Springer fiber can be expressed as an orbital integral. In fact, the orbital integrals occurring in the statement of the fundamental lemma can be expressed in terms of affine Springer fibers, and one can say that the fundamental lemma predicts some (totally unexpected) kind of relationship between affine Springer fibers for different groups.

One can try to prove the fundamental lemma by studying affine Springer fibers. Goresky, Kottwitz and MacPherson [31] have proved the fundamental lemma in the "equivaluation case", using their theorems about affine Springer fibers that we stated above (Theorem 3.17, Theorem 3.18). Laumon had the idea of using deformations of affine Springer fibers to make the problem more accessible, see [51], but still only obtained the result (for unitary groups) assuming the purity conjecture of Goresky, Kottwitz and MacPherson.

A break-through occurred with the work of Laumon and Ngô [53] who realized that the "Hitchin fibration" is a suitable global situation into which (slight modifications of) the relevant affine Springer fibers can be embedded. This provides a geometric interpretation of the theory of endoscopy. Heuristically, one gets a natural way to deform, and eventually "get rid of", the singularities. The work of Laumon and Ngô dealt with the simpler case of unitary groups, where the endoscopic groups in question are actually subgroups of the original group, so that it is easier than in the general case to relate the intervening geometric objects to each other. Recently, Ngô in his celebrated paper [61] was able to overcome the big remaining difficulties and to prove the fundamental lemma in the general case.

In addition to the original papers cited above we mention the surveys written by Dat [16], Laumon [52] and Ngô [60]. Neither of these includes the most recent and complete results by Ngô [61], but see the preprint [17] by Dat and Tuan.

## 4. Affine Deligne-Lusztig Varieties

### 4.1. Deligne-Lusztig Varieties

We start with a short reminder about usual Deligne-Lusztig varieties in order to put the theory described below into context. Let $k$ be an algebraic closure of the finite field $\mathbb{F}_q$, let $G$ be a connected reductive group over $\mathbb{F}_q$, let $B$ be a Borel subgroup defined over $\mathbb{F}_q$, and let $T \subseteq B$ be a maximal torus defined over $\mathbb{F}_q$. We denote by $\sigma$ the Frobenius morphism on $k$, $G(k)$, etc. Let $W$ be the absolute Weyl group of the pair $(G, T)$, i.e., the Weyl group for $T_k \subset G_k$. If $G$ is split, then $W$ is equal to the Weyl group "over $\mathbb{F}_q$", but here it is unnecessary to make this assumption. Recall the Bruhat decomposition $G(k) = \bigcup_{w \in W} B(k)wB(k)$.

**Definition 4.1 (Deligne-Lusztig,** [18]**).** The *Deligne-Lusztig variety* associated with $w \in W$ is the locally closed subvariety $X_w \subset G_k/B_k$ with

$$X_w(k) = \{g \in G(k);\ g^{-1}\sigma(g) \in B(k)wB(k)\}/B(k).$$

Given $g, h \in (G/B)(k) = G(k)/B(k)$, one says that the relative position $\mathrm{inv}(g, h)$ of $g$, $h$ is the unique element $w \in W$, such that $g^{-1}h \in B(k)wB(k)$. The latter condition should be understood as a condition on representatives of $g$, $h$ in $G(k)$, but is independent of the choice of representatives. With this notion, we can say that $X_w(k)$ is the set of all elements $g \in (G/B)(k)$ such that $g$ and $\sigma(g)$ have relative position $w$.

**Example 4.2.** If $w = \mathrm{id}$ is the identity element, then $X_{\mathrm{id}} = (G/B)(\mathbb{F}_q)$ is the set of $\mathbb{F}_q$-rational points in $G_k/B_k$, i.e., the set of fix points of $\sigma$.

**Example 4.3.** If $G = \mathrm{GL}_n$, then we can identify $G(k)/B(k)$ with the set of full flags of subvector spaces in $k^n$. We identify the Weyl group with the subgroup of permutation matrices, and hence with the symmetric group $S_n$ on $n$ letters. The relative position of flags $\mathscr{F}_\bullet$, $\mathscr{G}_\bullet$ can be described as follows: It is the unique permutation $\gamma \in S_n$ such that for all $i$, $j$,

$$\dim(\mathscr{F}_i \cap \mathscr{G}_j) = \#\{1 \leq l \leq j;\ \gamma(l) \leq i\}.$$

We record some foundational properties of Deligne-Lusztig varieties:

**Proposition 4.4.** *Let $w \in W$.*

1. *The Deligne-Lusztig variety $X_w$ is smooth and of dimension $\ell(w)$, the length of $w$.*
2. *The closure $\overline{X}_w$ of $X_w$ in $G_k/B_k$ is normal, and*
$$\overline{X}_w = \bigcup_{v \leq w} X_v,$$
*where $\leq$ denotes the Bruhat order in $W$.*
3. *The Deligne-Lusztig variety $X_w$ is connected if and only if $w$ is not contained in a $\sigma$-stable standard parabolic subgroup of $W$.*

In fact, generalizing 3., one can easily give a formula for the number of connected components of a Deligne-Lusztig variety.

*Proof.* See [18], [10], [25]. □

Deligne-Lusztig varieties play an important role in the representation theory of finite groups of Lie type, i.e., of groups of the form $G(\mathbb{F}_q)$, where $G$ is as above. The reason is that $G(\mathbb{F}_q)$ acts on $X_w$, and hence on its cohomology. Deligne and Lusztig [18] have shown that all irreducible representations of $G(\mathbb{F}_q)$ can be realized inside the ($\ell$-adic) cohomology of Deligne-Lusztig varieties, with suitable $G(\mathbb{F}_q)$-equivariant local systems as coefficients.

Deligne-Lusztig varieties also occur in many other situations. As one example we mention the results of C.-F. Yu and the author [29] which show that all "Kottwitz-Rapoport" strata that are entirely contained in the supersingular locus of a Siegel modular variety with Iwahori level structure are disjoint unions of copies of a Deligne-Lusztig variety.

**Remark 4.5.** There is a natural relationship between the Springer representation and the representations associated with Deligne-Lusztig varieties. This was first proved by Kazhdan; for details see the appendix of the paper [42] of Kazhdan and Varshavsky.

### 4.2. $\sigma$-Conjugacy Classes

Now and for the following sections we fix a finite field $\mathbb{F}_q$, and let $k$ be an algebraic closure of $\mathbb{F}_q$. The Frobenius $\sigma\colon x \mapsto x^q$ acts on $k$, and also (on the coefficients) on $L = k((\epsilon))$: $\sigma(\sum a_i \epsilon^i) = \sum a_i^q \epsilon^i$. We write $F = \mathbb{F}_q((\epsilon))$, the fixed field of $\sigma$ in $L$. As usual, we fix an algebraic group $G$ over $\mathbb{F}_q$. We assume, since that is the case we will consider below, that $G$ is a *split* connected reductive group (see Kottwitz' paper for the classification of $\sigma$-conjugacy classes in the general case).

Before we come to the definition of affine Deligne-Lusztig varieties, we discuss Kottwitz' classification of $\sigma$-conjugacy classes in $G(L)$. The (right) action of $G(L)$ on itself by $\sigma$-conjugation is given by $h \cdot g = g^{-1} h \sigma(g)$. Correspondingly, the $\sigma$-conjugacy class of $b$ in $G(L)$ is the subset $\{g^{-1} b \sigma(b);\ g \in G(L)\}$. We denote by $B(G)$ the set of $\sigma$-conjugacy classes in $G(L)$.

If we consider $G(k)$ instead of $G(L)$, then the situation is considerably simpler: Lang's theorem (see e. g. [11], Theorem 16.3) says that $G(k)$ is a single $\sigma$-conjugacy class. In fact, this statement is of crucial importance for all three points of Proposition 4.4.

The set $B(G)$ of $\sigma$-conjugacy classes was described by Kottwitz [44], [45]. A simple invariant of a $\sigma$-conjugacy class is the connected component of the loop group $G(L)$ it lies in. In other words, the map $\kappa\colon G(L) \to \pi_1(G)$ factors through a map $\kappa\colon B(G) \to \pi_1(G)$ (sometimes called the Kottwitz map).

A more interesting invariant of a $\sigma$-conjugacy class is its Newton vector, an element in $X_*(A)_{\mathbb{Q}}/W$. We will not give its definition here (see [44], [45]; see Example 4.6 for the case of $\mathrm{GL}_n$). For practical purposes, the following description is often good enough, however:

The restriction $N_GT(L) \to B(G)$ of the natural map from $G(L)$ to $B(G)$ factors through the extended affine Weyl group $\widetilde{W} = N_GT(L)/T(\mathcal{O})$. This follows from a variant of Lang's theorem. The resulting map $\widetilde{W} \to B(G)$ is surjective (this is implicit in Kottwitz' classification, see e. g. [27] Corollary 7.2.2). Now if $w \in \widetilde{W}$, its Newton vector $\nu$ can be computed as follows. Let $n$ be the order of the finite Weyl group part of $w$, i.e., the order of the image of $w$ under the projection $\widetilde{W} = X_*(A) \rtimes W \to W$. Then $w^n = \epsilon^\lambda$ for some translation element $\lambda \in X_*(A)$, and $\nu = \frac{1}{n}\lambda \in X_*(A)_\mathbb{Q}/W$. The resulting map $B(G) \to X_*(A)_\mathbb{Q}/W$ is called the *Newton map*.

Kottwitz shows that combining the Newton map with the map $\kappa$, one obtains an injection

$$B(G) \to X_*(A)_\mathbb{Q}/W \times \pi_1(G).$$

In the special case that the derived group $G^{\mathrm{der}}$ is simply connected, the connected component can be recovered from the Newton vector, so that the Newton map is injective. For instance this is true for $\mathrm{GL}_n$ and $\mathrm{SL}_n$, but not for $\mathrm{PGL}_n$. We sometimes identify the quotient $X_*(A)_\mathbb{A}/W$ with the dominant chamber $X_*(A)_{\mathbb{Q},+}$ of rational coweights $\lambda$ such that $\langle \alpha, \lambda\rangle \geq 0$ for all roots $\alpha$, and consider Newton vectors as elements of the latter.

**Example 4.6.** Let us consider the case $G = \mathrm{GL}_n$. Every $\sigma$-conjugacy class contains a representative $b$ of the following form: $b$ is a block diagonal matrix, and each block has the form

$$\begin{pmatrix} 0 & \epsilon^{k_i+1} I_{k_i'} \\ \epsilon^{k_i} I_{n_i-k_i'} & 0 \end{pmatrix} \in \mathrm{GL}_{n_i}(L).$$

Here $n = \sum n_i$, $k_i, k_i' \in \mathbb{Z}$, $0 \leq k_i < n$. The Newton vector of $b$ is the composite of the Newton vectors of the single block, and the Newton vector of each block is $(k_i + \frac{k_i'}{n_i}, \dots, k_i + \frac{k_i'}{n_i})$ (where the tuple has $n_i$ entries). This representative is called the standard representative in [27] 7.2.

This shows that the set of elements in $X_*(A)_{\mathbb{Q},+}$ (which we can identify with the set of $n$-tuples of rational numbers in descending order) is the subset of sequences

$$a_1 = \cdots = a_{i_1} > a_{i_1+1} = \cdots = a_{i_1+i_2} > a_{i_1+i_2+1} \cdots > a_{i_1+\cdots i_r+1} = \cdots = a_n$$

that satisfy the integrality condition $i_\nu a_{i_1+\cdots+i_{\nu-1}+1} \in \mathbb{Z}$ for each $1 \leq \nu \leq r+1$ (with $i_{r+1} = n - i_1 - \cdots - i_r$).

Given $(a_1, \dots, a_n) \in X_*(A)_{\mathbb{Q},+}$, we can view the $a_i$ as the slopes of the *Newton polygon* attached to $b$. (One usually orders the slopes in ascending order, so that the Newton polygon is the lower convex hull of the points $(0,0)$ and $(i, \sum_{j=n-i+1}^n a_j)$, $i = 1, \dots, n$.) The integrality condition says that the break points of this polygon should have integer coefficients.

The classification of $\sigma$-conjugacy classes in $\mathrm{GL}_n$ is the same as the classification of isocrystals (due to Dieudonné/Manin). More precisely, an isocrystal is a pair $(V, \Phi)$ consisting of a finite-dimensional $L$-vector space $V$ and a $\sigma$-linear bijection $\Phi$ (i.e., $\Phi$ is additive, $\Phi(av) = \sigma(a)v$ for all $a \in L$, $v \in V$, and $\Phi$ is bijective). Choosing a basis of $V$, we can write $\Phi = b\sigma$, $b \in \mathrm{GL}_n(L)$, $n = \dim V$. A change of basis corresponds to $\sigma$-conjugating $b$. See for instance Demazure's book [19], ch. IV.

Given $b \in G(L)$, its $\sigma$-centralizer is the algebraic group $J_b$ over $F$ with

$$J_b(F) = \{g \in G(L);\ g^{-1}b\sigma(g) = b\}.$$

In fact, $J_b$ is an inner twist of the Levi subgroup $\mathrm{Cent}_G(\nu)$, the centralizer of the Newton vector of $\nu$. See [44] 6.5.

**Definition 4.7 (**[47]**).** Let $b \in G(L)$. The *defect of* $b$ is the difference

$$\mathrm{def}(b) = \mathrm{rk}_F(G) - \mathrm{rk}_F(J_b)$$

of the ($F$-)rank of $G$ and the $F$-rank of the $\sigma$-centralizer $J_b$.

**Example 4.8.** If $b = 1$, then $J_b = G$ and $\mathrm{def}(b) = 0$. On the other hand, suppose that $G = \mathrm{GL}_n$, and let $b$ be the generator of the group of length 0 elements in $\widetilde{W}$ with Newton vector $(\frac{1}{n}, \dots, \frac{1}{n})$. Then $J_b = D^\times_{\frac{1}{n}}$, the group of units of the central division algebra over $F$ with invariant $\frac{1}{n}$. In this case, $\mathrm{def}(b) = n - 1$.

Kottwitz [47] has proved that the defect of an element $b \in G(L)$ can also be expressed in a way which is close to Chai's conjectural formula [14] for the dimension of Newton strata in Shimura varieties, and to Rapoport's conjectural formula ([64] Conj. 5.10) for the dimension of affine Deligne-Lusztig varieties in the affine Grassmannian (Theorem 4.17 below).

To simplify the discussion, we assume here that the derived group $G^{\mathrm{der}}$ is simply connected. We have an exact sequence

$$1 \to G^{\mathrm{der}} \to G \to D \to 1,$$

where $D := G/G^{\mathrm{der}}$ is a split torus, and obtain an exact sequence of character groups

$$0 \to X^*(D) \to X^*(A) \to X^*(A \cap G^{\mathrm{der}}) \to 0.$$

We lift the fundamental weights in $X^*(A \cap G^{\mathrm{der}})$ to elements $\omega_i$, $i = 1, \dots, l$ in $X^*(A)$, and in addition choose a basis $\omega_{l+1}, \dots, \omega_n$ of $X^*(D)$. Then the characters $\omega_1, \dots, \omega_n$ form a $\mathbb{Z}$-basis of $X^*(A)$.

**Example 4.9.** If $G = \mathrm{GL}_n$, then $G^{\mathrm{der}} = \mathrm{SL}_n$ is simply connected, and a possible choice of the $\omega_i$ is

$$\omega_i = (1^{(i)}, 0^{(n-i)}) \in \mathbb{Z}^n = X^*(A), \quad i = 1, \dots, n,$$

the notation meaning that 1 is repeated $i$ times, and 0 is repeated $n - i$ times.

**Proposition 4.10.** *Assume that the derived group $G^{\mathrm{der}}$ is simply connected, and let $b \in G(L)$ with Newton vector $\nu_b$. Then*

$$\mathrm{def}(b) = 2\sum_{i=1}^{n} \mathrm{fr}(\langle \omega_i, \nu_b \rangle),$$

*where for any rational number* $\alpha$*,* $\mathrm{fr}(\alpha) \in [0,1)$ *denotes its fractional part.*

Finally, we make the following important definition:

**Definition 4.11.** A $\sigma$-conjugacy class in $G(L)$ is called *basic*, if the following equivalent conditions are satisfied:

1. The Newton vector $\nu$ is central, i.e., lies in the image of $X_*(Z)_{\mathbb{Q}}$, where $Z \subseteq G$ is the center of $G$.
2. The $\sigma$-conjugacy class can be represented by an element $\tau \in \widetilde{W}$ with $\ell(\tau) = 0$.

We call an element $b \in G(L)$ basic, if its $\sigma$-conjugacy class is basic.

More precisely, one can show that the restriction of the map $\widetilde{W} \to B(G)$ to the set $\Omega_G$ of elements of length 0 is a bijection from $\Omega_G$ to the set of basic $\sigma$-conjugacy classes ([45] 7.5, [27] Lemma 7.2.1). Looking at $\sigma$-conjugacy classes from the point of view of Newton strata in the special fiber of a Shimura variety, the basic locus is the unique closed Newton stratum. In the case of the Siegel modular variety, for instance, this is just the supersingular locus.

### 4.3. Affine Deligne-Lusztig Varieties: The Hyperspecial Case

Similarly as for usual Deligne-Lusztig varieties, we want to consider all elements $g$ which map under a "Lang map" to a fixed double coset. When we consider the affine Grassmannian, we look at $G(\mathcal{O})$-cosets, which by the Cartan decomposition (Theorem 2.17) are parameterized by the set $X_*(A)_+$ of dominant coweights. Furthermore, in the affine context one has to consider generalizations of the Lang map, i.e., we consider maps of the form $g \mapsto g^{-1}b\sigma(g)$ for an element $b \in G(L)$.

**Definition 4.12.** The *affine Deligne-Lusztig variety* $X_\mu(b)$ in the affine Grassmannian associated with $b \in G(L)$ and $\mu \in X_*(A)_+$ is given by

$$X_\mu(b)(k) = \{g \in G(L);\ g^{-1}b\sigma(g) \in G(\mathcal{O})\epsilon^\mu G(\mathcal{O})\}/G(\mathcal{O}).$$

We can view this definition as a $\sigma$-linear variant of affine Springer fibers, and as it turns out, there are several analogies between the two theories. For instance, see the discussion of the dimension formula (Theorem 4.17) below. Of course, there are also many differences, and an important difference is that the definition above includes a parameter $\mu$: While in the case of affine Springer fibers we always considered the set of $g$ such that $g^{-1}bg \in G(\mathcal{O})$ (or rather the Lie algebra version of this), here we consider this relationship for an arbitrary $G(\mathcal{O})$-double coset. Furthermore, while in the case of affine Springer fibers the element $\gamma$, which corresponds to the $b$ above, was regular semisimple in the most interesting cases, in the case of affine Deligne-Lusztig varieties the most interesting case is where $b$ is a basic element, i.e., up to $\sigma$-conjugacy, $b$ is a representative of a length 0 element of $\widetilde{W}$.

The subset $X_\mu(b)(k)$ is locally closed in $\mathcal{G}rass(k)$, so it inherits the structure of a (reduced) sub-ind-scheme. In fact, $X_\mu(b)$ is a scheme locally of finite type over $k$; see Corollary 5.5 in the paper [36] by Hartl and Viehmann. The key point here is that points $x$ in the building such that the distance from $x$ to $b\sigma(x)$ is bounded (the bound being given by $\mu$) have bounded distance to the building of $J_b$ over a finite unramified extension of $F$; this was proved by Rapoport and Zink [67]. Compare the corresponding fact for affine Springer fibers, where every point has bounded distance to $X_*(A_\gamma)$ (Proposition 3.8 1.). Usually $X_\mu(b)$ has infinitely many irreducible components:

**Proposition 4.13.** *Assume that $G$ is simple and of adjoint type. Let $b \in G(L)$, $\mu \in X_*(A)$ with $X_\mu(b) \neq \emptyset$. The following are equivalent:*

1. *$X_\mu(b)$ is of finite type over $k$.*
2. *The element $b$ is superbasic, i.e., no $\sigma$-conjugate of $b$ is contained in a proper Levi subgroup of $G$.*
3. *$G = \mathrm{PGL}_n$ and the Newton vector of $b$ has the form $(\frac{r}{n}, \dots, \frac{r}{n})$, $r \in \mathbb{Z}$ coprime to $n$.*

*Proof.* The implication 2.⇒ 1., which is the most difficult one, follows from Viehmann's detailed study of the superbasic case [77]. The equivalence of 2. and 3. is explained in [26] 5.9. There it is also shown that $b$ is superbasic if and only if $J_b$ is anisotropic. If this is not the case, then the $J_b$-action on $X_\mu(b)$ shows that $X_\mu(b)$ contains points of arbitrarily high distance to the origin (in the sense of the building), and hence cannot be of finite type. □

**Remark 4.14.**
1. Multiplication by $g^{-1} \in G(L)$ induces an isomorphism between $X_\mu(b)$ and $X_\mu(g^{-1}b\sigma(g))$. Since we are only interested in affine Deligne-Lusztig varieties up to isomorphism, we are free to replace $b$ by another representative of its $\sigma$-conjugacy class. The $\sigma$-centralizer $J_b(F)$ of $b$ acts on $X_\mu(b)$.
2. We may have $X_\mu(b) = \emptyset$. It is one of the basic questions, when this happens. The reason that usual Deligne-Lusztig varieties are always non-empty is that the Lang map $g \mapsto g^{-1}\sigma(g)$ is a surjection $G(k) \to G(k)$: All elements of $G(k)$ are $\sigma$-conjugate to the identity element. Therefore in the classical case there is no need to introduce the parameter $b$ which we see above. On the other hand, the Lang map $G(L) \to G(L)$ is not surjective. In fact, we have seen above that $G(L)$ consists of many $\sigma$-conjugacy classes. The usual proof for the surjectivity of the Lang map fails in the setting of ind-schemes: although the differential of the Lang map $g \mapsto g^{-1}\sigma(g)$ is an isomorphism, one cannot conclude that the map itself is "étale".
3. One can obviously generalize the definition to cover other parahoric subgroups of $G(L)$. One obtains affine Deligne-Lusztig varieties inside (partial) affine flag varieties. We will consider the case of the Iwahori subgroup in detail in the following section.
4. There is a $p$-adic variant, where $L$ is replaced by the completion of the maximal unramified extension of $\mathbb{Q}_p$. In this case, the same definition gives an *affine Deligne-Lusztig set.* Although one still uses the term *affine Deligne-Lusztig variety* in the $p$-adic situation, this is not really justified. One does not have a $k$-ind-scheme structure on the quotient $G(L)/I$, and hence there

is no variety structure on $X_\mu(b)$. On the other hand, this case is particularly interesting because of its connection to the theory of moduli spaces of $p$-divisible groups and Shimura varieties (see 4.9 below). In fact, in some cases one obtains a variety structure on the affine Deligne-Lusztig set, induced from the scheme structure of the corresponding moduli space.
5. We mention that in [57], Lusztig has considered a different analogue of usual Deligne-Lusztig varieties in an "affine" context (his varieties are infinite-dimensional, they have a pro-structure rather than an ind-structure).
6. We mention in passing that it is also interesting to replace $\sigma$ by other morphisms. See the papers by Baranovsky and Ginzburg [3] and by Caruso [13] for two other choices of $\sigma$. The first one is related to the study of conjugacy classes in Kac-Moody groups, the latter one is motivated by the theory of Breuil, Kisin and others about the classification of finite flat group schemes.

**Example 4.15.** If $G = \mathrm{SL}_2$, then $X_*(A)_+ = \{(a, -a) \in \mathbb{Z}^2;\ a \geq 0\}$, and an element of the affine Grassmannian is in the $G(\mathcal{O})$-orbit corresponding to $(a, -a)$ if and only if it, seen as a vertex in the Bruhat-Tits building over $L$, has distance $a$ to the rational building (i.e., the building over $F$). In particular, we find a description of affine Deligne-Lusztig varieties $X_\mu(1)$ which is completely analogous to the description of affine Springer fibers for $\mathrm{SL}_2$ in Section 3.4.1.

There is the following criterion for non-emptiness of $X_\mu(b)$:

**Theorem 4.16.** *Let $b \in G(L)$ with Newton vector $\nu \in X_*(A)_{\mathbb{Q},+}$, and let $\mu \in X_*(A)$ be dominant. Then*

$$X_\mu(b) \neq \emptyset \quad \Longleftrightarrow \quad \kappa_G(b) = \mu \text{ and } \nu \leq \mu,$$

*where we denote the image of $\mu$ in $\pi_1(G)$ again by $\mu$, and where $\nu \leq \mu$ means by definition that $\mu - \nu$ is a non-negative linear combination of simple coroots.*

The implication $\Rightarrow$ is called Mazur's inequality; for $\mathrm{GL}_n$ the statement above boils down to a version of an inequality considered by Mazur in the study of $p$-adic estimates of the number of points over a finite field of certain algebraic varieties. It was proved by Rapoport and Richartz [65] for general $G$. The converse, accordingly called the "converse to Mazur's inequality" was proved only recently. It was conjectured to hold, and proved for $\mathrm{GL}_n$ and $\mathrm{GSp}_{2g}$, by Kottwitz and Rapoport [48]. C. Lucarelli [55] proved the theorem for classical groups, and finally Gashi [23] proved it for the exceptional groups (and even in the quasi-split case). See also Kottwitz' paper [46].

**Theorem 4.17.** *Let $b \in G(L)$ with Newton vector $\nu_b \in X_*(A)_{\mathbb{Q},+}$, and let $\mu \in X_*(A)$ be dominant. Assume that $X_\mu(b) \neq \emptyset$. Then*

$$\dim X_\mu(b) = \langle \rho, \mu - \nu_b \rangle - \frac{1}{2} \operatorname{def}(b).$$

Recall that we denote by $\rho$ half the sum of the positive roots. The defect in the dimension formula should be seen as a correction term, and should be compared with the term $\frac{1}{2}(\operatorname{rk}\mathfrak{g} - \dim \mathfrak{a}^w)$ in the dimension formula for affine Springer fibers (Proposition 3.8), compare [47], in particular (1.9.1).

The formula for the dimension of $X_\mu(b)$ was conjectured by Rapoport in [64] Conj. 5.10. and reformulated in the current form by Kottwitz [47], using the notion of defect. In [26], the proof of the dimension formula is reduced to the superbasic case, i.e., to the case where no $\sigma$-conjugate of $b$ is contained in a proper Levi subgroup of $G$. Subsequently the formula was proved in the superbasic case by Viehmann [77].

To reduce the proof of the dimension formula to the superbasic case, one has to compare the affine Deligne-Lusztig varieties $X_\mu^M(b)$ and $X_\mu^G(b)$ for a Levi subgroup $A \subseteq M \subseteq G$ and $b \in M(L)$. If $P = MN$ is a parabolic subgroup, then there is a bijection $P(L)/P(\mathfrak{o}) \cong G(L)/K$, and this defines a map

$$\alpha\colon G(L)/K \cong P(L)/P(\mathfrak{o}) \longrightarrow M(L)/M(\mathfrak{o})$$

from the affine Grassmannian for $G$ to the affine Grassmannian for $M$. This map is not a morphism of ind-schemes, but for any connected component $Y$ of the affine Grassmannian for $M$, the restriction of $\alpha$ to $\alpha^{-1}(Y)$ is a morphism of ind-schemes. This map can be used to relate the affine Deligne-Lusztig varieties for $M$ and for $G$; see [26] 5.6.

It is expected that in the hyperspecial case, all affine Deligne-Lusztig varieties are equidimensional. This has been proved in the two extreme cases:

If $b = \epsilon^\nu$ is a translation element, then Proposition 2.17.1 in [26] shows that $X_\mu(b)$ is equidimensional. The proof relies on a result proved by Mirković and Vilonen as part of their proof of the geometric Satake isomorphism. More precisely, their results about the intersection cohomology of intersections of $U(L)$- and $K$-orbits imply that these intersections are equidimensional. For a proof of the relevant fact in positive characteristic, which is what one needs in our situation, see the paper [62] by Ngô and Polo.

On the other hand, if $b$ is basic, then it was proved by Hartl and Viehmann in [36] that $X_\mu(b)$ is equidimensional; see Section 4.10.

To prove equidimensionality in general, one might want to apply the strategy of the proof of the dimension formula, that is to reduce to the (super-)basic case. However, to carry out this reduction, a better understanding of the restriction of the map $\alpha$ to $X_\mu^G(b)$ would be needed.

A general result about the relationship between affine Deligne-Lusztig varieties for $G$ and Levi subgroups of $G$ is the Hodge-Newton decomposition. Let $M_b$ be the centralizer of the Newton vector $\nu$ of $b$. There is a unique standard parabolic subgroup $P_b = M_b N_b$ with Levi subgroup $M_b$ (and unipotent radical $N_b$). Denote by $A_{P_b}$ the identity component of the center of $M$, and let

$$\mathfrak{a}_{P_b}^+ = \{x \in X_*(A_{P_b}) \otimes \mathbb{R};\ \langle \alpha, x\rangle > 0 \text{ for every root } \alpha \text{ of } A_{P_b} \text{ in } N_b\}.$$

Replacing $b$ by a $\sigma$-conjugate, we may and will assume that $b \in M_b(L)$, $b$ is basic with respect to $M_b$, and $\kappa_{M_b}(b) \in X_*(A_{P_b})_{\mathbb{R}}$ actually lies in $\mathfrak{a}_{P_b}^+$. Given any standard parabolic subgroup $P \subseteq G$, we use analogous notation as for $P_b$: $P = MN$, $A_P$, etc. Then we have

**Theorem 4.18 (**[46]**;** [78] **Theorem 1).** *Let $\mu \in X_*(A)$ be dominant, and let $b$, $M_b$ as above. Let $P = MN \subseteq G$ be a standard parabolic subgroup with $P_b \subseteq P$. If $\kappa_M(b) = \mu$, then the inclusion $X_\mu^M(b) \to X_\mu^G(b)$ is an isomorphism.*

See also the paper [59] by Mantovan and Viehmann for a generalization to unramified groups.

It is hard to determine the set of connected components of an affine Deligne-Lusztig variety $X_\mu(b)$. The problem is that in general the group $J_b(F)$ does not act transitively on the set of connected components, see Viehmann's paper [78], Section 3, for an example. As Viehmann shows in loc. cit., the situation is better if instead one considers the following variant: Given $\mu$, define

$$X_{\leq\mu}(b) = \bigcup_{\lambda\leq\mu} X_\lambda(b).$$

This is a closed subscheme of the affine Grassmannian (equipped with the reduced scheme structure). A priori, it could be bigger than the closure of $X_\mu(b)$ in $\mathcal{G}rass_G$, though. For $b$ basic, Hartl and Viehmann [36] show that $X_{\leq\mu}(b)$ is equal to the closure of $X_\mu(b)$. For these "closed affine Deligne-Lusztig varieties" one has

**Theorem 4.19 (Viehmann** [78] **Theorem 2).** *Suppose that the data $G$, $\mu$, $b$ are indecomposable with respect to a Hodge-Newton decomposition, i.e., there is no standard $P \supseteq P_b$ with $\kappa_M(b) = \mu$. Assume that $G$ is simple.*

1. *If $\epsilon^\mu$ is central in $G$, and $b$ is $\sigma$-conjugate to $\epsilon^\mu$, then*
$$X_\mu(b) = X_{\leq\mu}(b) \cong J_b(F)/(J_b(F) \cap G(\mathcal{O})) \cong G(F)/G(\mathcal{O}_F)$$
*is discrete.*
2. *Assume that we are not in the situation of 1. Then $\kappa_M(b) \neq \mu$ for all proper standard parabolic subgroups $P = MN \subsetneq G$ with $b \in M(L)$, and $\kappa_G$ induces a bijection*
$$\pi_0(X_{\leq\mu}(b)) \cong \pi_1(G).$$

*The group $J_b(F)$ acts transitively on $\pi_0(X_{\leq\mu}(b))$.*

### 4.4. Affine Deligne-Lusztig Varieties: The Iwahori Case

Now we come to the Iwahori case. Because of the Iwahori-Bruhat decomposition $G(L) = \bigcup_{x\in\widetilde{W}} IxI$ (Theorem 2.18) we associate affine Deligne-Lusztig varieties inside the affine flag variety to elements $b \in G(L)$ and $x \in \widetilde{W}$:

**Definition 4.20.** The *affine Deligne-Lusztig variety* $X_x(b)$ in the affine flag variety associated with $b \in G(L)$ and $x \in \widetilde{W}$ is given by

$$X_x(b)(k) = \{g \in G(L);\ g^{-1}b\sigma(g) \in IxI\}/I.$$

The same remarks as in the case of the affine Grassmannian apply. In fact, in the Iwahori case it is much harder (and not yet completely settled) to give a criterion for which $X_x(b)$ are non-empty, and a closed formula for their dimensions.

Note that $X_x(b) = \emptyset$ whenever $x$ and $b$ do not lie in the same connected component of $G(L)$. Since for each $x \in \widetilde{W}$ we have a unique basic $\sigma$-conjugacy class in the same connected component as $x$, there is a unique basic $\sigma$-conjugacy class for which $X_x(b)$ can possibly be non-empty. Therefore, as long as we talk only about basic $\sigma$-conjugacy classes, $x$ practically determines $b$, and below we sometimes assume implicitly that $x$ and $b$ are in the same connected component of $G(L)$.

**Example 4.21.** Let us discuss the case of $G = \mathrm{SL}_2$, $b = 1$. For $\mathrm{SL}_2$, the situation is particularly simple. For instance, every element in the affine Weyl group of $\mathrm{SL}_2$ has a unique reduced expression, and there are only two elements of any given length $> 0$. What are the Schubert cells $IxI$ which can contain an element of the form $g^{-1}\sigma(g)$? Fix $gI \in G(L)/I$ with $g \neq \sigma(g)$. We consider $gI$ as an alcove in the Bruhat-Tits building of $\mathrm{SL}_2$, and denote by $d$ the distance from $gI$ to the rational building (normalizing the distance so that an alcove has distance 0 to the rational building if and only if it is contained in there). Clearly, $\sigma(g)I$ also has distance $d$ to the rational building, and because the rational building is equal to the subcomplex of $\sigma$-fix points in the building over $L$, one sees that the distance from $gI$ to $\sigma(g)I$ is $2d-1$. This implies that the element $x \in \widetilde{W}$ with $g^{-1}\sigma(g) \in IxI$ has length $2d-1$, and shows that $X_x(1) = \emptyset$ if $\ell(x)$ is different from 0 and even. It is not hard to show that on the other hand all $X_x(1)$ for $x$ of odd length are non-empty, and that in this case $\dim X_x(1) = \frac{1}{2}(\ell(x)+1)$. For a more detailed consideration along these lines see Reuman's PhD thesis [68].

To simplify some of the statements below, we assume from now on, for the rest of Section 4.4, that the Dynkin diagram of $G$ is connected. The (non-)emptiness of affine Deligne-Lusztig varieties for simply connected groups of rank 2 is illustrated in Section 4.5 below. Looking at these pictures, or rather at the picture for a fixed $b$ (cf. [26], [27]), one notices that the behavior is more complicated and more difficult to describe close to the walls of the finite root system. The following definition was made by Reuman in order to describe the "good" region.

**Definition 4.22.** Let $x \in \widetilde{W}$. We say that $x$ lies in the *shrunken Weyl chambers*, if for every finite root $\alpha$, $U_\alpha(L) \cap xIx^{-1} \neq U_\alpha(L) \cap I$.

In other words, $x$ lies in the shrunken Weyl chambers, if for every finite root $\alpha$ there exists an affine root hyperplane parallel to $\{\alpha = 0\}$ which separates $x$ and the base alcove. We sometimes call the complement of the shrunken chambers the *critical strips*. See the left hand side picture in Figure 2.

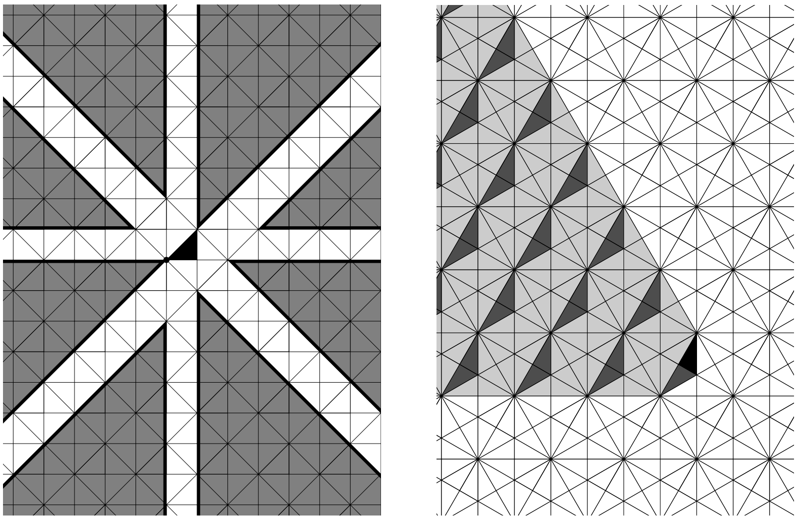

FIGURE 2. Left: The shrunken Weyl chambers (gray) in the root system of type $C_2$. Right: The set of $P$-alcoves for the group of type $G_2$ and $P = {}^{s_1s_2s_1}(B \cap Bs_2B)$.

We define maps from the extended affine Weyl group $\widetilde{W}$ to the finite Weyl group $W$ as follows:

$\eta_1 \colon \widetilde{W} = X_*(T) \rtimes W \to W,$ the projection

$\eta_2 \colon \widetilde{W} \to W,$ where $\eta_2(x)$ is the unique element $v \in W$ such that $v^{-1}x \in {}^S\widetilde{W}$

$\eta(x) = \eta_2(x)^{-1}\eta_1(x)\eta_2(x).$

Let $x \in \widetilde{W}$, and let $b$ be a representative of the unique basic $\sigma$-conjugacy class corresponding to the connected component of $x$. We define the *virtual dimension* (of $X_x(b)$):

$$d(x) = \frac{1}{2}(\ell(x) + \ell(\eta(x)) - \operatorname{def}(b)).$$

The following conjecture which extends a conjecture made by D. Reuman [69] gives a very simple "closed formula" for non-emptiness and dimension of affine Deligne-Lusztig varieties for $b$ basic and $x$ in the shrunken Weyl chambers. (Note that the conjecture as it stands does not extend to all $x$. See Conjecture 4.28 below for a more precise, but more technical conjecture about non-emptiness.)

**Conjecture 4.23** ([27] **Conj. 9.4.1 (a)).** *Let $b \in G(L)$ be basic. Assume that $x \in \widetilde{W}$ lies in the shrunken Weyl chambers. Then $X_x(b) \neq \emptyset$ if and only if $\kappa_G(x) = \kappa_G(b)$ and $\eta(x) \in W \setminus \bigcup_{T \subsetneq S} W_T$. In this case,*

$$\dim X_x(b) = d(x)$$

Note that this statement easily implies the dimension formula for affine Deligne-Lusztig varieties in the affine Grassmannian. Several partial results towards this conjecture have been obtained. In [27], one direction is proved; in fact, there is the following slightly stronger result which also covers most of the critical strips:

**Theorem 4.24** ([27], **Proposition 9.4.4).** *Let $b$ be basic. Let $x \in \widetilde{W}$, say $x = \epsilon^\lambda v$, $v \in W$. Assume that $\lambda \neq \nu_b$ and that $\eta(x) \in \bigcup_{T \subsetneq S} W_T$. Then $X_x(b) = \emptyset$.*

As in the hyperspecial case, the key point of the theorem is to relate certain affine Deligne-Lusztig varieties for $G$ to Deligne-Lusztig varieties for Levi subgroups $M$. Since the group of connected components of the loop group of $M$ is much larger than the one for $G$, the trivial condition that $b$ and $x$ must belong to the same connected component becomes much stronger, and yields an obstruction for affine Deligne-Lusztig varieties to be non-empty. To single out the elements $x \in \widetilde{W}$, where this can be done, we need the notion of $P$-alcove introduced in loc. cit.; see the right hand side picture of Figure 2 for an example.

In the sequel, we consider parabolic subgroups $P \subseteq G$. They need not be standard, i.e., we do not require that $B \subseteq P$, but we only consider *semi-standard* parabolic subgroups, i.e., we ask that $A \subseteq P$. We denote by $P = MN$ the Levi decomposition of such a subgroup; here $N$ denotes the unipotent radical of $P$, and $M$ is the unique Levi subgroup of $P$ which contains $A$. Then the (extended affine, or finite) Weyl group of $M$ is contained in the (extended affine, or finite) Weyl group of $G$.

**Definition 4.25.** Let $P = MN \subseteq G$ be a semi-standard parabolic subgroup. An element $x \in \widetilde{W}$ is called a *$P$-alcove*, if it satisfies the following conditions:

1. $x \in \widetilde{W}_M$, the extended affine Weyl group of $M$,
2. $x(I \cap N(L))x^{-1} \subseteq I \cap N(L)$.

For $P$-alcoves, one has a "Hodge-Newton decomposition" (see above and [46], [59] for analogues in the hyperspecial case):

**Theorem 4.26** ([27] **Theorem 2.1.4).** *Suppose that $x$ is a $P$-alcove for $P = MN \supseteq A$. If $X_x(b) \neq \emptyset$, then the $\sigma$-conjugacy class of $b$ meets $M(L)$. Now assume that $b \in M(L)$. Then the closed immersion $X_x^M(b) \to X_x(b)$ induces a bijection*

$$J_b^M(F)\backslash X_x^M(b) \xrightarrow{\cong} J_b(F)\backslash X_x(b).$$

*Here $X_x(b)$ denotes the affine Deligne-Lusztig variety for $M$, and $J_b^M$ denotes the $\sigma$-centralizer of $b$ in $M$.*

This is deduced easily from the following, slightly more technical statement, which shows that $x$ being a $P$-alcove is a strong requirement from the point of view of $\sigma$-conjugacy classes occurring inside $IxI$.

**Theorem 4.27** ([27], **Theorem 2.1.2).** *Let $x$ be a $P$-alcove for the semi-standard parabolic subgroup $P = MN \subset G$. Then every element of $IxI$ is $\sigma$-conjugate under $I$ to an element of $I_M x I_M$, where $I_M = I \cap M$.*

These results, together with experimental evidence, lead to the following conjecture:

**Conjecture 4.28** ([27], **Conj. 9.3.2).** *Let $b$ be basic with Newton vector $\nu_b$, and let $x \in \widetilde{W}$. Then $X_x(b)$ is empty if and only if there exists a semi-standard parabolic subgroup $P = MN \subseteq G$, such that $x$ is a $P$-alcove, and $\kappa_M(x) \neq \kappa_M(\nu_b)$.*

On the other hand, still assuming that $b$ is basic, in [28] X. He and the author prove non-emptiness of $X_x(b)$ using the "reduction method of Deligne and Lusztig" (see [18], proof of Theorem 1.6, or [28]) and combinatorial considerations about the affine Weyl group for all elements $x$ that are sufficiently far from the walls and which are expected to give rise to a non-empty $X_x(b)$. More precisely, let us denote by $\rho^\vee$ the sum of all fundamental coweights, and by $\theta$ the largest root.

**Definition 4.29.** An element $\mu \in X_*(A)$ is said to lie in the *very shrunken* Weyl chambers, if

$$|\langle \mu, \alpha \rangle| \geq \langle \rho^\vee, \theta \rangle + 2$$

for every root $\alpha$.

We then have the following theorem.

**Theorem 4.30** ([28]). *Let let $b$ be basic, let $x \in \widetilde{W}$ be in the same connected component of $G(L)$ as $b$, and write $x = t^\mu w$.*

1. *If $\mu$ is regular, or $\eta_2(x) = w_0$, the longest element of $W$, then* $\dim X_x(b) \leq$ *$d(x)$.*
2. *Assume that $\eta(x) \in W \setminus \bigcup_{T \subsetneq S} W_T$. If $\mu$ is in the very shrunken Weyl chambers or $\eta_2(x) = w_0$, then $X_x(b) \neq \emptyset$.*
3. *Let $G$ be a classical group, and let $x \in W_a$ be an element of the affine Weyl group such that $\eta(x) \in W \setminus \bigcup_{T \subsetneq S} W_T$. If $\mu$ is in the very shrunken Weyl chambers or $\eta_2(x) = w_0$, then* $\dim X_x(1) = d(x)$.

A crucial ingredient for part 3. is a theorem of He [37] about conjugacy classes in affine Weyl groups. E. Beazley has independently obtained similar results, using the reduction method of Deligne and Lusztig and results by Geck and Pfeiffer about conjugacy classes in finite Weyl groups.

If $b$ is not basic, then because of the experimental evidence we expect the following, see [27] Conj. 9.4.1 (b). To simplify the statement, let us assume that $b$ is in $\widetilde{W}$, and is of minimal length among all the elements representing this $\sigma$-conjugacy class. Let $x \in \widetilde{W}$ be in the same connected component of $G(L)$ as $b$.

- If $\ell(x)$ is small (with respect to $\ell(b)$), then $X_x(b) = \emptyset$.
- If $\ell(x)$ is large (with respect to $\ell(b)$), then $X_x(b) \neq \emptyset$ if and only if $X_x(b_{\text{basic}}) \neq \emptyset$, where $b_{\text{basic}}$ represents the unique basic $\sigma$-conjugacy class in the same connected component as $x$. In this case, the dimension of the two affine Deligne-Lusztig varieties differs by a constant (depending on $b$, but not on $x$).

It is not easy to give precise bounds for what "small" and "large" should mean. (For the first, one gets approximate information by considering the projection to the affine Grassmannian.) This question can be viewed as the problem of finding a suitable analogue of Mazur's inequality in the Iwahori case. It is hard to describe the pattern of non-emptiness for $x$ of length close to $\ell(b)$. For abundant examples, see the figures for the rank 2 case given in the next section.

One can also study the question of non-emptiness from the slightly different point of view where one fixes $x$, and asks for the set of $b$'s which give a non-empty affine Deligne-Lusztig variety. See Beazley's paper [6] for an analysis of the case of $\mathrm{SL}_3$ from this standpoint.

In the Iwahori case, affine Deligne-Lusztig varieties are not equidimensional in general. An example with $G = \mathrm{SL}_4$ is given in [28].

### 4.5. The Rank 2 Case

There are several ways of computing, in specific cases, whether an affine Deligne-Lusztig variety is non-empty, and what its dimension is. Let us illustrate one approach to eliminate the Frobenius morphism from the problem and hence to reduce it to a purely combinatorial statement, in the case $b = 1$. By definition, $X_x(1) \neq \emptyset$ if and only if there exists $g \in G(L)$ such that $g^{-1}\sigma(g) \in IxI$. Now we decompose $G(L) = \bigcup_{w\in\widetilde{W}} IwI$, and we see that the existence of $g$ as before is equivalent to the existence of $w \in \widetilde{W}$ and $i \in I$ with $w^{-1}i^{-1}\sigma(i)w \cap IxI \neq \emptyset$. Now the group $I$ is a single $\sigma$-conjugacy class: $I = \{i^{-1}\sigma(i);\ i \in I\}$. Therefore the existence of $g$ is equivalent to the existence of $i' \in I$ with $w^{-1}i'w \cap IxI \neq \emptyset$, or in other words:

$$X_x(1) \neq \emptyset \iff x \in Iw^{-1}IwI \text{ for some } w \in \widetilde{W}.$$

For given $x$ and $w$, the condition on the right hand side can easily be translated into a combinatorial statement about the Bruhat-Tits building or the affine Weyl group, respectively. See [26] 6,7, and [27] 10–13, for a detailed discussion. In the case of rank 2 it is in principle feasible to do such computations "by hand"; see Reuman's papers [68], [69]. Using a computer program, one can assemble a large number of examples.

Here we give examples for simply connected groups of type $A_2$ (i.e., $\mathrm{SL}_3$), $C_2$ (i.e., $\mathrm{Sp}_4$) and $G_2$, for $b = 1$. We identify the affine Weyl group $W_a$ with the set of alcoves in the standard apartment, i.e., with the set of small triangles in the figures below. The base alcove is marked by a thick border.

We denote the $\sigma$-conjugacy classes by letters, according to the tables given below. The letters associated with $\sigma$-conjugacy classes which meet an Iwahori double coset are printed inside the corresponding alcove. To save space, sequences of letters without gaps are abbreviated as follows: ABCDE is abbreviated to A-E, etc. When denoting elements in $\widetilde{W}$ as products in the generators, we write $s_{021}$ instead of $s_0s_2s_1$, etc.

**4.6. Type $A_2$**

In this case, the set of $\sigma$-conjugacy classes is well-known, and we just need to say which $\sigma$-conjugacy classes occur in the figure, and how they are named.

| | $b =$ | $\overline{\nu}_b =$ | | $b =$ | $\overline{\nu}_b =$ |
|---|---|---|---|---|---|
| A | $\epsilon^{(0,0,0)} = 1$ | $(0,0,0)$ | Q | $\epsilon^{(4,1,-5)}$ | $(4,1,-5)$ |
| B | $\epsilon^{(1,0,-1)}$ | $(1,0,-1)$ | R | $\epsilon^{(3,2,-5)}$ | $(3,2,-5)$ |
| C | $\epsilon^{(2,0,-2)}$ | $(2,0,-2)$ | S | $\epsilon^{(6,0,-6)}$ | $(6,0,-6)$ |
| D | $\epsilon^{(2,-1,-1)}$ | $(2,-1,-1)$ | T | $\epsilon^{(6,-1,-5)}$ | $(6,-1,-5)$ |
| E | $\epsilon^{(1,1,-2)}$ | $(1,1,-2)$ | U | $\epsilon^{(6,-2,-4)}$ | $(6,-2,-4)$ |
| F | $\epsilon^{(3,0,-3)}$ | $(3,0,-3)$ | V | $\epsilon^{(6,-3,-3)}$ | $(6,-3,-3)$ |
| G | $\epsilon^{(3,-1,-2)}$ | $(3,-1,-2)$ | W | $\epsilon^{(5,1,-6)}$ | $(5,1,-6)$ |
| H | $\epsilon^{(2,1,-3)}$ | $(2,1,-3)$ | X | $\epsilon^{(4,2,-6)}$ | $(4,2,-6)$ |
| I | $\epsilon^{(4,0,-4)}$ | $(4,0,-4)$ | Y | $\epsilon^{(3,3,-6)}$ | $(3,3,-6)$ |
| J | $\epsilon^{(4,-1,-3)}$ | $(4,-1,-3)$ | a | $s_{021}$ | $(1,-\frac{1}{2},-\frac{1}{2})$ |
| K | $\epsilon^{(4,-2,-2)}$ | $(4,-2,-2)$ | b | $s_{012}$ | $(\frac{1}{2},\frac{1}{2},-1)$ |
| L | $\epsilon^{(3,1,-4)}$ | $(3,1,-4)$ | c | $s_{021021021}$ | $(3,-\frac{3}{2},-\frac{3}{2})$ |
| M | $\epsilon^{(2,2,-4)}$ | $(2,2,-4)$ | d | $s_{012012012}$ | $(\frac{3}{2},\frac{3}{2},-3)$ |
| N | $\epsilon^{(5,0,-5)}$ | $(5,0,-5)$ | e | $s_{021021021021021}$ | $(5,-\frac{5}{2},-\frac{5}{2})$ |
| O | $\epsilon^{(5,-1,-4)}$ | $(5,-1,-4)$ | f | $s_{012012012012012}$ | $(\frac{5}{2},\frac{5}{2},-5)$ |
| P | $\epsilon^{(5,-2,-3)}$ | $(5,-2,-3)$ | | | |

**4.7. Type $C_2$**

Again, it is well-known what the $\sigma$-conjugacy classes are, so we just list those which we consider, and under which names they appear.

| | $b =$ | $\overline{\nu}_b =$ | | $b =$ | $\overline{\nu}_b =$ |
|---|---|---|---|---|---|
| A | $\epsilon^{(0,0)} = 1$ | $(0,0)$ | K | $\epsilon^{(4,0)}$ | $(4,0)$ |
| B | $\epsilon^{(1,0)}$ | $(1,0)$ | L | $\epsilon^{(4,1)}$ | $(4,1)$ |
| C | $\epsilon^{(1,1)}$ | $(1,1)$ | M | $\epsilon^{(4,2)}$ | $(4,2)$ |
| D | $\epsilon^{(2,0)}$ | $(2,0)$ | a | $\epsilon^{(6,5)}$ | $(6,5)$ |
| E | $\epsilon^{(2,1)}$ | $(2,1)$ | b | $\epsilon^{(6,6)}$ | $(6,6)$ |
| F | $\epsilon^{(2,2)}$ | $(2,2)$ | c | $s_{012}$ | $(\frac{1}{2},\frac{1}{2})$ |
| G | $\epsilon^{(3,0)}$ | $(3,0)$ | d | $s_{012012012}$ | $(\frac{3}{2},\frac{3}{2})$ |
| H | $\epsilon^{(3,1)}$ | $(3,1)$ | e | $s_{012012012012012}$ | $(\frac{5}{2},\frac{5}{2})$ |
| I | $\epsilon^{(3,2)}$ | $(3,2)$ | f | $s_{012012012012012012012}$ | $(\frac{7}{2},\frac{7}{2})$ |
| J | $\epsilon^{(3,3)}$ | $(3,3)$ | g | $s_{012012012012012012012012012}$ | $(\frac{9}{2},\frac{9}{2})$ |

**4.8. Type $G_2$**

The set $B(G)$ is the union of the set of dominant translation elements, and the following two families, each coming from one of the two standard parabolic subgroups:

$$\{n \cdot (-\frac{1}{6},-\frac{1}{6},\frac{1}{3});\ n \in \mathbb{Z}_{>0}\}, \quad \{n \cdot (0,-\frac{1}{2},\frac{1}{2});\ n \in \mathbb{Z}_{>0}\}.$$

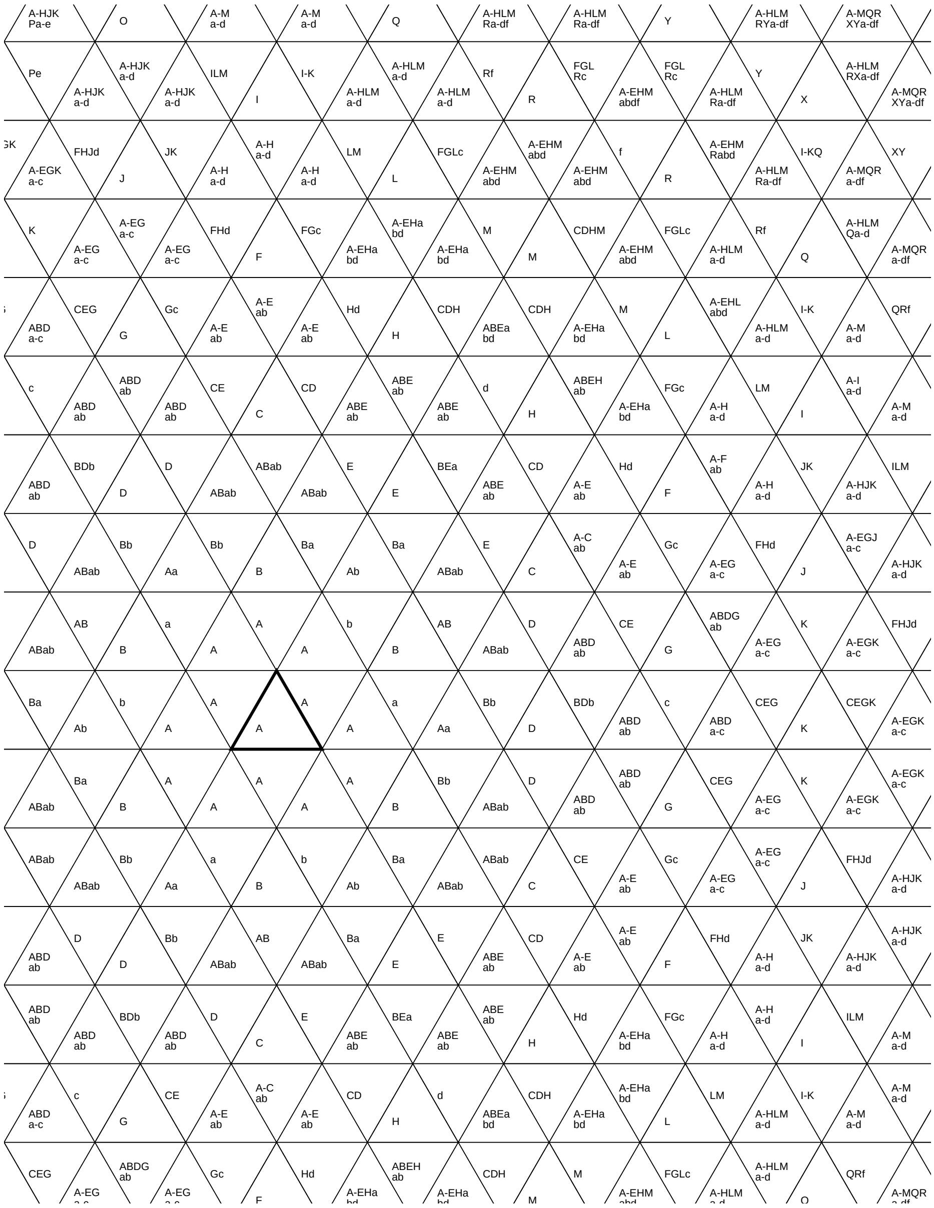


FIGURE 3. Dimensions of affine Deligne-Lusztig varieties, type $A_2$, $b = 1$.

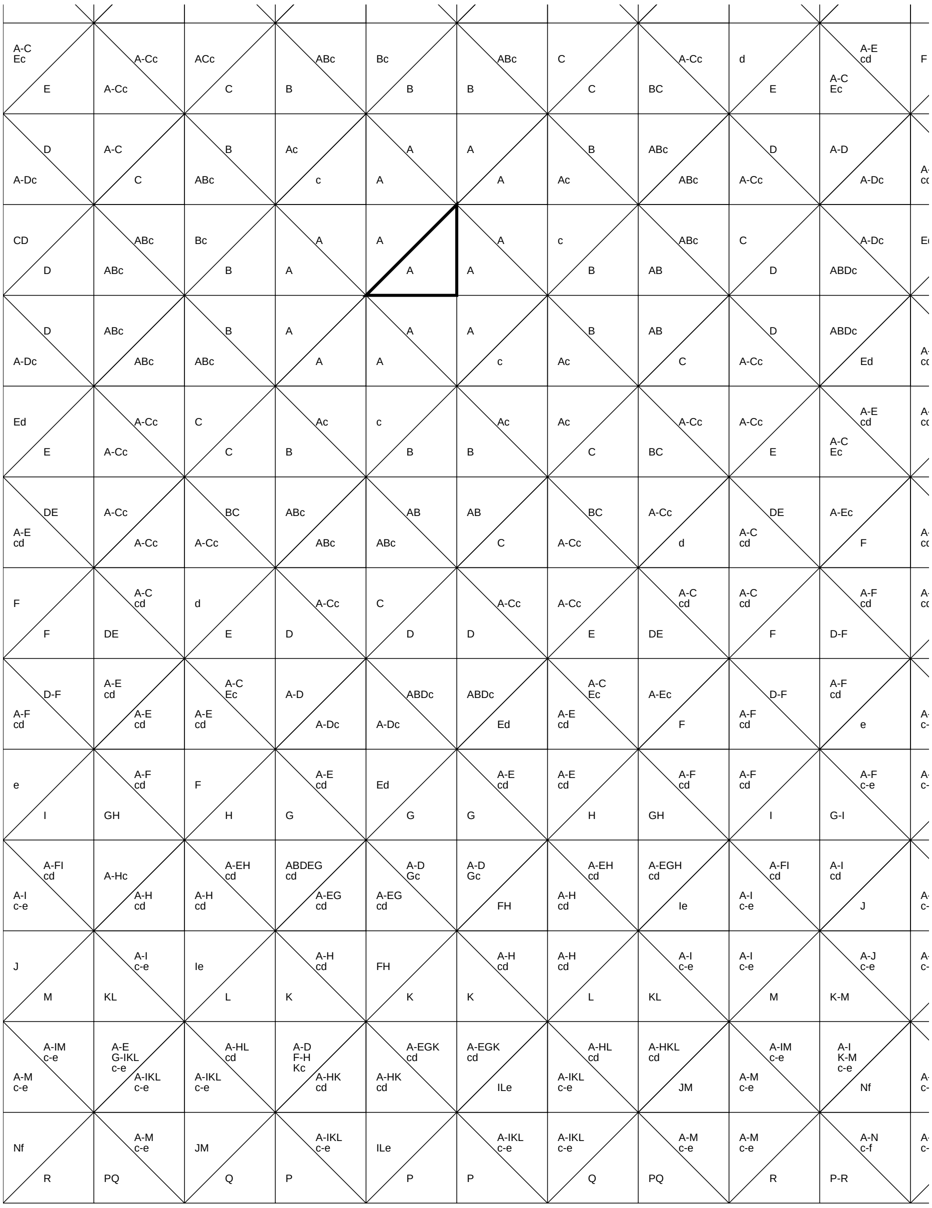


FIGURE 4. Dimensions of affine Deligne-Lusztig varieties, type $C_2$, $b = 1$.

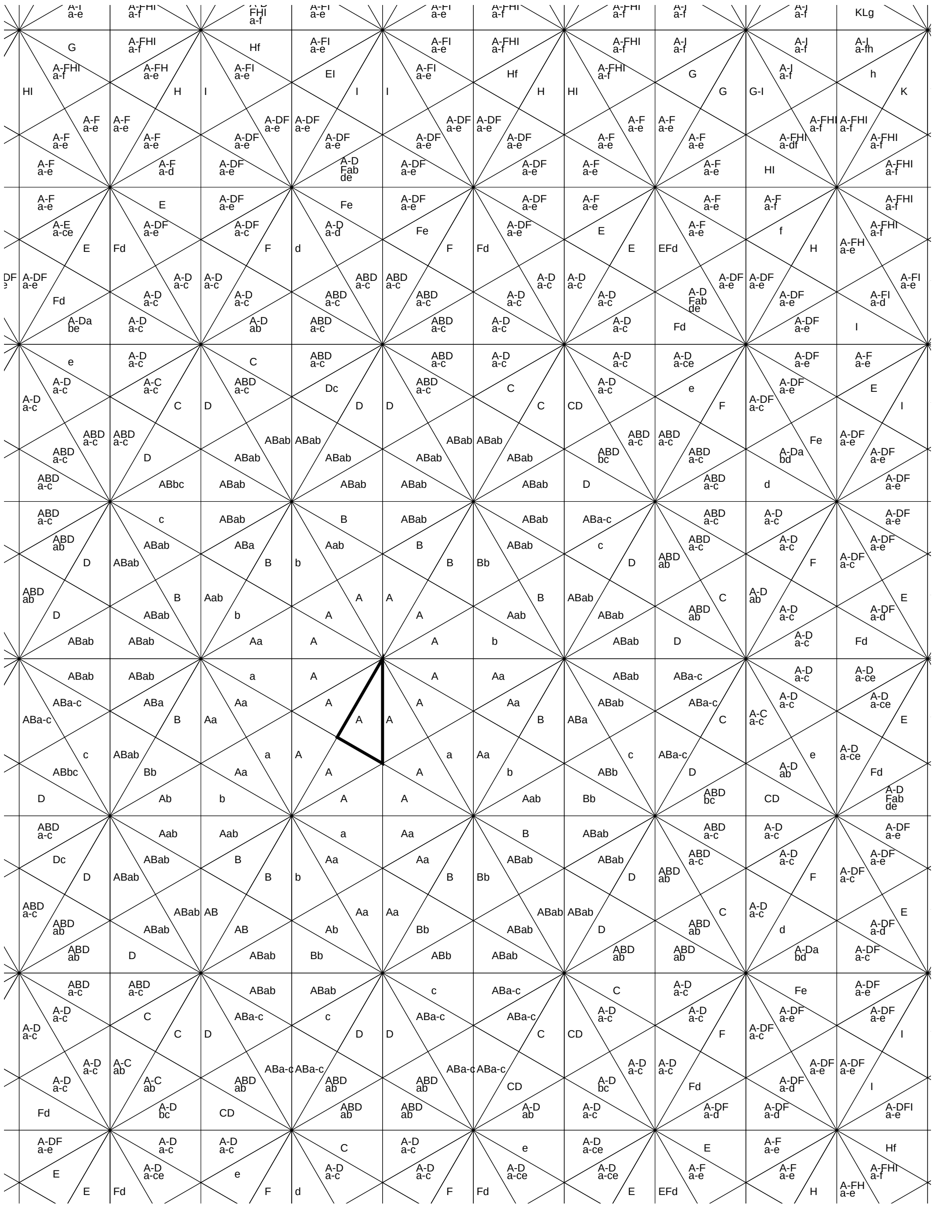

Figure 5. Dimensions of affine Deligne-Lusztig varieties, type $G_2$, $b = 1$.

To find the alcove corresponding to a certain reduced expression, recall that with our normalization the shortest edge of the alcoves is of type 0, the medium edge is of type 1, and the longest edge is of type 2.

Here is the list of $\sigma$-conjugacy classes considered for the figure.

| | $b =$ | $\overline{\nu}_b =$ | | $b =$ | $\overline{\nu}_b =$ |
|---|---|---|---|---|---|
| A | $\epsilon^{(0,0,0)} = 1$ | $(0,0,0)$ | K | $\epsilon^{(-1,-2,3)}$ | $(-1,-2,3)$ |
| B | $\epsilon^{(-\frac{1}{3},-\frac{1}{3},\frac{2}{3})}$ | $(-\frac{1}{3},-\frac{1}{3},\frac{2}{3})$ | L | $\epsilon^{(-\frac{1}{3},-\frac{7}{3},\frac{8}{3})}$ | $(-\frac{1}{3},-\frac{7}{3},\frac{8}{3})$ |
| C | $\epsilon^{(-\frac{2}{3},-\frac{2}{3},\frac{4}{3})}$ | $(-\frac{2}{3},-\frac{2}{3},\frac{4}{3})$ | a | $s_{021}$ | $(-\frac{1}{6},-\frac{1}{6},\frac{1}{3})$ |
| D | $\epsilon^{(0,-1,1)}$ | $(0,-1,1)$ | b | $s_{02121}$ | $(0,-\frac{1}{2},\frac{1}{2})$ |
| E | $\epsilon^{(-1,-1,2)}$ | $(-1,-1,2)$ | c | $s_{021210212}$ | $(-\frac{1}{2},-\frac{1}{2},1)$ |
| F | $\epsilon^{(-\frac{1}{3},-\frac{4}{3},\frac{5}{3})}$ | $(-\frac{1}{3},-\frac{4}{3},\frac{5}{3})$ | d | $s_{021210212102121}$ | $(0,-\frac{3}{2},\frac{3}{2})$ |
| G | $\epsilon^{(-\frac{4}{3},-\frac{4}{3},\frac{8}{3})}$ | $(-\frac{4}{3},-\frac{4}{3},\frac{8}{3})$ | e | $s_{021212021210212}$ | $(-\frac{5}{6},-\frac{5}{6},\frac{5}{3})$ |
| H | $\epsilon^{(-\frac{2}{3},-\frac{5}{3},\frac{7}{3})}$ | $(-\frac{2}{3},-\frac{5}{3},\frac{7}{3})$ | f | $s_{021212021210212021212}$ | $(-\frac{7}{6},-\frac{7}{6},\frac{7}{3})$ |
| I | $\epsilon^{(0,-2,2)}$ | $(0,-2,2)$ | g | $s_{021210212102121021210212102121}$ | $(0,-\frac{5}{2},\frac{5}{2})$ |
| J | $\epsilon^{(-\frac{5}{3},-\frac{5}{3},\frac{10}{3})}$ | $(-\frac{5}{3},-\frac{5}{3},\frac{10}{3})$ | h | $s_{021212021210212021212021212}$ | $(-\frac{3}{2},-\frac{3}{2},3)$ |

### 4.9. Relationship to Shimura Varieties

Affine Deligne-Lusztig varieties are related to the reduction of certain Shimura varieties, or more directly to moduli spaces of $p$-divisible groups. To establish this relationship, we consider the $p$-adic variant of affine Deligne-Lusztig varieties, cf. Remark 4.14 4. The relation relies on Dieudonné theory (see for instance [19]), which classifies $p$-divisible groups over a perfect field in terms of their Dieudonné modules. A Dieudonné module is a free module of finite rank over the ring of Witt vectors $W$ together with a $\sigma$-linear operator $F$ (Frobenius) and a $\sigma^{-1}$-linear operator $V$ (Verschiebung) such that $FV = VF = p$ (so that $V$ is uniquely determined by $F$).

Now fix a $p$-divisible group $\mathbb{X}$ over $k$, and denote by $M$ its Dieudonné module, and by $N = M \otimes_W W[\frac{1}{p}]$ its rational Dieudonné-module, or *isocrystal*. We fix a basis of $M$ over $W$ and write $F$ as $b\sigma$, $b \in \mathrm{GL}_n(W[\frac{1}{p}])$, where $n = \mathrm{rk}_W\, M$. Lattices inside $N$ which are stable under $F$ and $V$ correspond to quasi-isogenies $\mathbb{X} \to X$ of $p$-divisible groups over $k$. A lattice $\mathscr{L} = gM$, $g \in \mathrm{GL}_n(W[\frac{1}{p}])$ is stable under $F$ and $V$ if and only if

$$p\mathscr{L} \subseteq F\mathscr{L} \subseteq \mathscr{L},$$

i.e., $g^{-1}b\sigma(g) \in G(\mathcal{O})\epsilon^{\mu}G(\mathcal{O})$ for some $\mu$ of the form $(1,\dots,1,0,\dots,0)$. In other words, $\mu$ is a minuscule dominant coweight. Since $\mu$ must have the same image as $b$ under the Kottwitz map $\kappa_G$, it is determined uniquely by $\mathbb{X}$. Therefore we can identify the set of $k$-valued points of the moduli space of quasi-isogenies attached to $\mathbb{X}$ with the affine Deligne-Lusztig set attached to $\mathrm{GL}_n$, $b$ and $\mu$ over $L = \widehat{\mathbb{Q}}_p^{\mathrm{un}}$. See the book [66] by Rapoport and Zink for more information about these moduli spaces, which are often called Rapoport-Zink spaces nowadays. One can also consider variants for other groups, associated with EL- or PEL-data; see loc. cit. See Viehmann's papers [79], [80] for results about the structure of such moduli spaces of $p$-divisible groups from this point of view. For instance, Viehmann determines the sets of connected and irreducible components, and the dimensions.

Similarly, one obtains a relationship to Shimura varieties, or more precisely to the Newton strata in the special fiber of the corresponding moduli space of abelian varieties. Restricting to a Newton stratum corresponds to fixing an isogeny type of $p$-divisible groups, i.e., to choosing $b$. Roughly speaking, the Newton stratum splits up, up to a finite morphism, as a product of a truncated Rapoport-Zink space and a "central leaf". See Mantovan's paper [58] and [26], 5.10 for details and further references.

One can also consider the Iwahori case from this point of view. Again, choosing $b$ corresponds to fixing a Newton stratum (or to considering a Rapoport-Zink space instead of a moduli space of abelian varieties). The choice of $x \in \widetilde{W}$ corresponds to the choice of a Kottwitz-Rapoport stratum. The affine Deligne-Lusztig set $X_x(b)$ is related to the intersection of these two strata. For instance, $X_x(b) \neq \emptyset$ if and only of Newton stratum for $b$ and KR stratum for $x$ intersect (see Haines' survey [33] Proposition 12.6).

One can show using an algorithmic description of the non-emptiness question (see [27]) that the $p$-adic variant of $X_x(b)$ is non-empty if and only if the function field variant $X_x(b)$ is non-empty (to formulate this properly, we assume that $b \in \widetilde{W}$). In particular, all of the results above in this direction yield information about the intersections of Newton strata and Kottwitz-Rapoport strata and hence about the geometric structure of these moduli spaces of abelian varieties. This is used by Viehmann [81] to obtain results about Shimura varieties from considerations about the function field case. On the other hand, there is no good a priori notion of dimension for the $p$-adic affine Deligne-Lusztig sets. It seems, however, that once one has a reasonable dimension theory for spaces of this kind, then the dimensions of $X_x(b)$ should agree in the $p$-adic and function field case. In the supersingular case, the dimension of the affine Deligne-Lusztig variety and the corresponding intersection of a Newton and a Kottwitz-Rapoport stratum are expected to be equal; in general there should be a non-trivial central leaf which governs the difference between the affine Deligne-Lusztig variety and the intersection.

For many more details along these lines see the survey papers by Haines [33] and Rapoport [64].

### 4.10. Local Shtuka

In [36], Hartl and Viehmann relate affine Deligne-Lusztig varieties to deformations of local shtuka. One could call this the function field version of Section 4.9. In the function field case, the theory works in full generality (whereas in the context of $p$-divisible groups one is limited to minuscule cocharacters, and also has limitations on which groups one can consider). Using their theory, they prove that for basic $b$, all affine Deligne-Lusztig varieties $X_\mu(b) \subset \mathcal{G}rass$ are equidimensional, and that the closure of $X_\mu(b)$ is equal to $\bigcup_{\lambda \leq \mu} X_\lambda(b)$.

### 4.11. Cohomology of Affine Deligne-Lusztig Varieties

Consider an affine Deligne-Lusztig variety $X_w(b)$. The $\sigma$-centralizer $J_b$ acts on $X_w(b)$, and hence on its cohomology with compact support, and on its (Borel-Moore) homology. Since usual Deligne-Lusztig varieties are nowadays an indispensable tool in the representation theory of finite groups of Lie type, one expects that the representations of $J_b$ occurring in the homology of affine Deligne-Lusztig varieties are also of great interest. However, because the geometric properties in the affine case are so much harder to understand, at the moment not much is known about representation theoretic properties. Let us give an overview about the results obtained so far.

Zbarsky [82] considered the following case: $G = \mathrm{SL}_3$, $b = \epsilon^\nu$ where $\nu$ is dominant regular. In this case, $J_b = A(F) \cong \mathbb{Z}^2 \times A(\mathcal{O}_F)$. Zbarsky shows that the subgroup $A(\mathcal{O}_F)$ acts trivially on the Borel-Moore homology of $X_w(b)$, and that the action of $\mathbb{Z}^2$ corresponds to permutation of the homology spaces of disjoint closed subsets of $X_w(b)$. A strategy to show that the integral part of the torus acts trivially is to extend the action to an action of $A(\mathcal{O}_L)$. An action of the latter must be trivial because of a "homotopy argument"; therefore the action of the subgroup $A(\mathcal{O}_F)$ is a fortiori trivial. However, it is not possible to extend the action in this way in general. Zbarsky defines a stratification of $X_w(b)$ such that on each stratum the action extends to an action of the larger torus, which is enough to reach the desired conclusion.

The Iwahori case for $G = \mathrm{GL}_2$ has been worked out in detail by Ivanov [40]. In this case, one can determine the geometric structure of the affine Deligne-Lusztig varieties completely. They are disjoint unions of copies of a product of some affine space with the complement of finitely many points on a projective line. As a consequence, one reads off directly the (co-)homology groups (with constant coefficients), and by analyzing the action of $J_b$ on the set of connected components one can determine the representations of $J_b$ which one gets; see loc. cit. Ivanov identifies these representations in terms of compact inductions, and also analyzes them from the point of view of the Langlands classification. There are no non-trivial morphisms to supercuspidal representations.

As in the finite-dimensional case, in addition to the homology with constant coefficients, one should also consider coefficients in certain local systems, or in other words, one should consider the homology of certain coverings of these affine Deligne-Lusztig varieties. Finally we mention the results of He [38] who, at least for $G = \mathrm{PGL}_n$ and $G = \mathrm{PSp}_{2n}$ identifies a subset of $\widetilde{W}$ such that all representations in the homology of affine Deligne-Lusztig varieties occur already in the homology of affine Deligne-Lusztig varieties $X_x(b)$ with $x$ in this subset.

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