Workshop on prismatic cohomology

Essen, March 27-29, 2018

In this workshop we will study the prismatic cohomolgy of Bhatt and Scholze. They define the notion of a prism which is by definition a ring A together with a Frobenius lift and a Cartier divisor $D \subset \operatorname{Spec}(A)$ satisfying certain conditions. Examples are $(\mathbb{Z}_p[[u]], V(u-p))$ or the ring $(A_{\mathrm{inf}}(C), \operatorname{Spec}(O_C))$ for some complete algebraically closed extension C of $\mathbb{Q}_p$. Then they attach to each smooth formal scheme X over D functorially a commutative algebra object in the derived category $D(A)$ together with a Frobenius map. Its cohomology interpolates between de Rham cohomology of the mod-p-reduction of X, the étale cohomology of $X[1/p]$ with p-torsion coefficients, and the crystalline cohomology of X.

This construction generalizes and simplifies the construction made in the paper [BMS]. Moreover, it yields a mixed characteristic Cartier isomorphism (that was somewhat implicit in [BMS]). Finally, the prismatic approach also allows to reprove basic results from the theory of perfectoid spaces such as the almost purity theorem in an easier fashion.

We hope that the primary reference [BS] will be available shortly. Meanwhile, an extremely valuable reference are Bhatt's lecture notes [B] which suffice to prepare almost all of the talks. Besides the theory of [BS] they also contain a lot of background information. In a few places, the lecture notes leave out technical details.

Talk 1. δ-rings (75 minutes).

Introduce the notion of δ-rings and their basic properties, in particular the equivalence between perfect δ-rings and perfect rings of characteristic p, and the relation between δ-rings and divided power algebras.

[B, Lecture II], [B, VI, Lemma 2.1], [BS, §1]

Talk 2. Derived completion (60 minutes).

Recall the notion of derived completion following [SP, Tag 091N] and [B, Lecture III, §2].

Talk 3. Prisms (75 minutes).

Distinguished elements, prisms, rigidity, prismatic envelope, the equivalence between perfect prisms and perfectoid rings.

[B, Lectures III, IV], [B, V, Lemma 5.1; VI, Cor. 2.2], [BS, §2].

Talk 4. The prismatic site and the Hodge-Tate comparison map (60 minutes).

Define the prismatic site and construct the HT comparison map, which will be proved to be an isomorphism in the next talk. If possible recall the generalities on cohomology used here (Čech-Alexander complex).

[B, Lecture V], [BS, §3].

Talk 5. Crystalline and Hodge-Tate comparison theorems (75 minutes).

Prove the crystalline comparison and the Hodge-Tate comparison, which ultimately relies on the Cartier isomorphism in de Rham cohomology. If time permits, one could say a few words about the Cartier isomorphism.

[B, Lecture VI], [BS, §3].

Talk 6. Non-abelian derived functors (90 minutes).

Explain the construction of non-abelian derived functors as used in [B, Lecture VII], following for example [Lu, §5.5.8]. Here we need the notion of ∞-categories, which might not be familiar for everyone.

Talk 7. Derived prismatic cohomology (60 minutes).

Introduce derived prismatic cohomology and derived de Rham cohomology and their basic properties (derived Cartier isomorphism, derived Hodge-Tate comparison). Prove the de Rham comparison theorem.

[B, Lecture VII], [BS, §3].

Talk 8. Application to perfectoid rings (60 minutes).

Every Zariski closed set in a perfectoid space is strongly Zariski closed; André's flatness lemma for perfectoid rings, the almost purity theorem.

As a preparation, discuss in some detail the process of perfection and perfectoidization, which will also be needed in the next talk.

[B, Lecture VIII], [BS, §4].

Talk 9. Étale comparison (75 minutes).

We suggest to use the references to the arc topology of [BM] whenever possible.

[B, Lecture IX], [BS, §5].

Talk 10. Application: q-de Rahm cohomology (75 minutes).

Define q-crystalline cohomology and prove its comparison with prismatic cohomology and q-de Rham cohomology.

[B, Lectures X,XI], [BS, §6], [Sch].

References

[B] B. Bhatt: Prismatic cohomology (lecture notes) `http://www-personal.umich.edu/~bhattb/teaching/prismatic-columbia/`

[BM] B. Bhatt, A. Mathew: The arc topology, arXiv:1807.04725

[BMS] B. Bhatt, M. Morrow, P. Scholze: Integral p-adic Hodge theory, arXiv:1602.03148

[BS] B. Bhatt, P. Scholze: Prismatic cohomology (in preparation)

[Lu] J. Lurie: Higher topos theory, Annals of Mathematics Studies **170**

[Sch] P. Scholze: Canonical q-deformations in arithmetic geometry, Ann. Fac. Sci. Toulouse Math. (6) 26 (2017), no. 5, 1163–1192.

[SP] Stacks Project.