RESEARCH SEMINAR: COHOMOLOGY OF MODULAR CURVES AND THE LOCAL LANGLANDS CORRESPONDENCE FOR GL_2

SS 2010

Let $K/\mathbb{Q}_p$ be a finite field extension. The local Langlands correspondence predicts the existence of a bijection between (certain) representations of a p-adic group such as $GL_n(K)$ and (certain) Galois representations, or more precisely representations $W_K \to GL_n(E)$ of the Weil group (with coefficients in an algebraically closed field E of characteristic 0, such as $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$). It has been proved for the group GL_n by Harris and Taylor (and shortly thereafter in a simpler way by Henniart) in 1998.

The aim of the seminar is to learn something about the local Langlands correspondence for GL_2 over $\mathbb{Q}_p$, and in particular about the connection to the cohomology of modular curves. This case has been known for a long time, based on the work of Langlands, Deligne and Carayol, among others. See [Ca]. We will not prove the local Langlands correspondence even in this case. For a proof, see the book [BH] by Bushnell and Henniart.

If there is enough time, or maybe as a continuation some time, it would be desirable to put the results of the seminar into a broader perspective. One interesting topic would be the work of Harris and Taylor [HT]; although this is very technical, it would be a natural choice insofar as the core of the seminar in a sense is "Harris-Taylor for $GL_2(\mathbb{Q}_p)$". Another theme would be the p-adic Langlands program, where currently a lot of exciting development is going on; see [E], for instance.

This seminar is closely related to some of the previous research seminars (although it does not formally depend on them), notably to the seminars on automorphic forms and on quaternion algebras and Shimura curves. Here however, we will put the main focus on understanding how the geometry of the special fibers of modular curves can be exploited to obtain representation theoretic information. Furthermore, many of the tools studied in last semester's seminar on p-divisible groups are very useful here.

The main reference for most of the seminar will be Carayol's paper [Ca] (which focusses on the case of Shimura curves, rather than the classical modular curves, but still seems to be the best reference available—the case of modular curves is in fact due to Deligne [D1]). As a kind of guide, one can take §2 in Emerton's paper [E] (but note that there the roles of p and ℓ are switched).

Let us describe the content of the seminar in more detail. Denote by $\mathbb{A}_f$ the ring of finite adeles over $\mathbb{Q}$. For a compact open subgroup $K \subset GL_2(\mathbb{A}_f)$, let Y_K denote the (non-compact) modular curve with level structure K. We consider Y_K as an algebraic curve over $\mathbb{Q}$.

Fix an integer $k \geq 1$ (think of k as the weight of the modular forms to be considered). One attaches to k a certain local system $\mathcal{F}_k$ on Y_K (either over $\mathbb{C}$, or—as we will understand it— an ℓ-adic sheaf in the sense of the étale topology).

Denote by

$$H^1_{\mathrm{par}}(Y_K, \mathcal{F}_k) := \mathrm{Im}(H^1_c(Y_K, \mathcal{F}_k) \to H^1(Y_K, \mathcal{F}_k))$$

the "parabolic" (or cuspidal) cohomology. Taking the direct limit

$$H^1_{\mathrm{par}}(\mathcal{F}_k) := \varinjlim H^1_{\mathrm{par}}(Y_K, \mathcal{F}_k)$$

over all K, and tensoring by $\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$, we obtain a representation $H^1_{\mathrm{par}}(\mathcal{F})_{\overline{\mathbb{Q}}_\ell}$ of $GL_2(\mathbb{A}_f) \times G_{\mathbb{Q}}$. Here $G_{\mathbb{Q}}$ denotes the absolute Galois group of $\mathbb{Q}$.

Theorem 1 (Eichler-Shimura theory). *This $GL_2(\mathbb{A}_f) \times G_{\mathbb{Q}}$-representation decomposes as*

$$H^1_{\mathrm{par}}(\mathcal{F}_k)_{\overline{\mathbb{Q}}_\ell} = \bigoplus_\pi \sigma(\pi) \otimes \pi, \tag{1}$$

where the sum runs over all cuspidal automorphic representations (satisfying a condition pertaining to k at the infinite place), or equivalently over all cuspidal newforms of weight k. For each π, $\sigma(\pi)$ is a 2-dimensional $G_{\mathbb{Q}}$-representation.

To state the next results, note that we can write an admissible irreducible representation π of $GL_2(\mathbb{A})$ as a restricted tensor product $\pi = \bigotimes'_v \pi_v$ of local components; in particular, for each prime p, we obtain a representation π_p of $GL_2(\mathbb{Q}_p)$.

The following theorem gives a description of the local Langlands correspondence in terms of the representations $\sigma(\pi)$.

Theorem 2 (Deligne, Carayol; [Ca] Théorème (B)). *Let π be a cuspidal automorphic representation of $GL_2(\mathbb{A})$ (satisfying some condition at the infinite place). Then there exists a number field E and a strictly compatible system $(\sigma^\lambda(\pi))_\lambda$ of continuous E_λ-adic Galois representations of $G_{\mathbb{Q}}$ such that for all primes $p > 2$, $\lambda \nmid p$,*

$$\sigma^\lambda(\pi)_{|W_{\mathbb{Q}_p}} \cong \mathcal{L}(\pi_p).$$

Here $\mathcal{L}$ denotes the (suitably normalized) Langlands correspondence between representations of $GL_2(\mathbb{Q}_p)$ and representations of $W_{\mathbb{Q}_p}$ with coefficients in $\overline{\mathbb{Q}}_\ell$, where ℓ is the residue characteristic of λ.

We also note the following point of view: (We take as level structure K a principal congruence subgroup $K = K_{Np^n}$, $p \nmid N$.)

Theorem 3 (see [E], Theorem 2.3.1). *In the decomposition*

$$H^1_{\mathrm{par}}(\mathcal{F}_k)_{\overline{\mathbb{Q}}_\ell} = \bigoplus_\pi \sigma(\pi) \otimes \pi,$$

$\sigma(\pi)$ is the $G_{\mathbb{Q}}$-representation attached to the cuspidal automorphic representation (=cusp form) π in the sense of Deligne, i.e., for all $p \nmid N\ell$ the representation is unramified, and we have

$$\mathrm{charpol}(\mathrm{Frob}_p) = X^2 - a_p X + \chi(p)p^{k-1},$$

where a_p is the Fourier coefficient of the cusp form corresponding to π, and χ is the nebentypus character attached to this cusp form.

From this point of view, Carayol's results provide information about what happens at the places $p|N$. For $p = \ell$, the situation is more complicated, but has been solved by Saito [Sa] using Fontaine theory. See [E], in particular Theorem 2.5.1.

The main part of Carayol's paper is the detailed analysis of the representation $\sigma(\pi)$, or rather its restriction to the Weil group of some $\mathbb{Q}_p$, in terms of the geometry of Y_K.

Fix a cuspidal automorphic representation π (in a sense, this is the same thing as a cusp form). Via the above construction, π gives rise to a Galois representation $\sigma(\pi)$. By restriction, we obtain $\sigma_p := \sigma(\pi)_{W_{\mathbb{Q}_p}}$. The key point of Carayol's work is a good understanding of σ_p, which is achieved by studying the geometry of modular curves and their reduction.

To make this more precise, consider the following short exact sequences:

(2)
$$\begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & {}^s\sigma_1 & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & \sigma_1 & \longrightarrow & \sigma_p & \longrightarrow & \sigma_2 & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & \widehat{\sigma}_1 & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

In this diagram, the horizontal exact sequence comes from the exact sequence for the sheaves of vanishing cycles. The contribution from the vanishing cycle sheaves lies in σ_2, while σ_1 lives in the cohomology of the special fiber (of a suitable model of the modular curve). The special fiber is the union of a finite number of smooth curves intersecting in the finitely many supersingular points. In the vertical exact sequence, σ_1 is split up according to the part $\widehat{\sigma}_1$ coming from the cohomology of the normalization (i.e., the disjoint union of the irreducible components of the special fiber) and some part ${}^s\sigma_1$ supported only at the supersingular points.

Program

1. **Overview and motivation.**

2. **Admissible representations.** Basic notions of the relevant representation theory of $GL_2(\mathbb{Q}_p)$. Discuss parabolic induction. Define the notions *cuspidal*, *discrete series*, *special representation*, *principal series*. See [W] 2.1, 2.2 (and maybe briefly 2.3, 2.4), [BH] §2, [Ku].

3. **Galois representations.** Basic notions about Galois representations. Weil group. Weil-Deligne group. Compare representations of these three groups. Main references: [W] §3.1, [BH] §28; see also [E] 2.1.2, [Ku], [T].

4. **The local Langlands correspondence.** Statement of the local Langlands correspondence for GL_2, as precise as possible (at least the definition of ϵ-factors can be omitted). Explanation of the unramified case. References: [W] §4, [BH] §33 (and maybe §§34, 35), [Ku].

5. **Geometry of modular curves (2 talks).** Discuss the construction of integral models of modular curves via Drinfeld level structures, and the geometry of their reduction. In particular, we need to know in the following

(1) that the special fiber is the union of finitely many smooth curves, each two of which intersect precisely in the supersingular points,
(2) the Eichler-Shimura congruence relation,
(3) the structure of the supersingular locus (as in [Ca] 1.7; see [Ko] or [S] §5).

See [KM] (in particular §13), the summary in [S] §4, §6, and [Ca] §1.

6. **Eichler-Shimura theory.** Explain the construction of the local system $\mathcal{F}_k$, and prove the decomposition (1).

References: [Ca] §2, [L] Thm. 2.10, Prop. 2.11, see also [D2], Thm. 2.10.

7. **Decomposition of the representation σ_p.** Introduce the notion of vanishing cycle sheaves, and present some examples. In particular we need the exact sequence relating the cohomology of the generic and the special fiber, and the vanishing cycles. References: SGA 7, exp. I, [W] 5.4, cf. also [Ca] §4.

Construct the short exact sequences of diagram (2), [Ca] §4.

8. **Analysis of σ_1.** Using points 1., 2. of talk 5 one can analyze the representation $\widehat{\sigma}_1$. For instance, one obtains that if $\widehat{\sigma}_1 \neq 0$, then π_p is an irreducible principal series representation. See [Ca] §5. Cf. also [L].

Next, using point 3. of talk 5 about the structure of the supersingular locus, one can compute the representation ${}^s\sigma_1$ quite explicitly, [Ca] §6.

9. **Deformations of formal groups and the fundamental local representation (2 talks).** A key point to proceed further is to rewrite the stalks of the vanishing cycle sheaves at the supersingular points in terms of the completion of the local ring. This completion is the universal deformation ring of a deformation problem of formal groups, and can therefore be studied by deformation theory. From this point of view, one can bring the group $\overline{G}$, the group of units of the division quaternion algebra ramified exactly at p and ∞, into the play. The interplay between the representations of $\overline{G}$ and GL_2 as described by the Jacquet-Langlands correspondence allows one to understand the structure of the representation σ_2 in the above diagram.

See [Ca] §7 – §11.1, but note that the theory is simpler in our case. This connects nicely with the Thursday afternoon seminar of the previous semester. We use the Jacquet-Langlands correspondence as a black box.

10. **Outlook . . . if time permits.**

References

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