Prof. Dr. Ulrich Görtz **SS 2013**
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Algebraic Geometry 2, Exercises

Sheet 1, due April 19

Exercise 1

1. Show that the notions of *open immersion* and *closed immersion* are local on the target: If $f\colon X \to Y$ is a morphism of schemes, and $Y = \bigcup_i V_i$ is an open covering, then f is an open/closed immersion if and only if for all i, the induced morphism $f^{-1}(V_i) \to V_i$ is an open/closed immersion.
2. Show that the notions of *open immersion* and *closed immersion* are stable under composition.
3. Show that the notions of *open immersion* and *closed immersion* are stable under base change, i.e. if $f\colon X \to Y$ is an open/closed immersion and $Y' \to Y$ is a morphism, then the base change $X \times_Y Y' \to Y'$ is an open/closed immersion.

Exercise 2

Let $f\colon X \to Y$ be a morphism of schemes.

1. Let f be a closed immersion. Show that f is a monomorphism, i.e. that for all schemes S the map $X(S) \to Y(S)$ induced by f is injective.
2. Give an example of a monomorphism f which is a homeomorphism onto a closed subset of Y, but which is not a closed immersion.

Exercise 3

Let X be a scheme.

1. Let $Z \subseteq X$ be a locally closed subset of X. Prove that there exists a unique reduced subscheme of X with underlying topological space Z. (*Hint:* First reduce to the case that Z is closed in X, and then to the affine case.)
2. In particular, there exists a unique reduced closed subscheme of X with the same underlying topological space as X. We denote this closed subscheme by X_{red}. Let $f\colon X \to Y$ be a morphism of schemes. Show that f induces a morphism $f_{\mathrm{red}}\colon X_{\mathrm{red}} \to Y_{\mathrm{red}}$. (One can show that this construction is functorial, i.e. compatible with composition of morphisms.)
3. Let $f\colon X \to Y$ be a morphism of schemes, and assume that X is reduced. Prove that f factors through Y_{red}.