Prof. Dr. Ulrich Görtz **SS 2013**
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Algebraic Geometry 2, Exercises

Sheet 10, due June 21

Exercise 28

Let X be a scheme (or any ringed space).

1. Let F, G be contravariant functors on the category of $\mathscr{O}_X$-modules to itself. Assume that F and G are left exact, and consider a morphism $F \to G$ of functors, such that $F(\mathscr{F}) \to G(\mathscr{F})$ is an isomorphism for every free module of finite rank. Prove that $F(\mathscr{F}) \to G(\mathscr{F})$ is an isomorphism for every $\mathscr{O}_X$-module of finite presentation.

2. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite presentation, and let $\mathscr{G}$ be any $\mathscr{O}_X$-module. Prove that for all $x \in X$, the canonical homomorphism
$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$
is an isomorphism.

Exercise 29

Let R be a noetherian ring. An R-module M is called projective, if the functor $N \mapsto \mathrm{Hom}_R(M, -)$ from the category of R-modules to itself is exact. Let M be a finitely generated R-module. Prove that the following are equivalent.

(i) M is a locally free R-module.

(ii) There exists a finitely generated R-module N such that $M \oplus N$ is a free R-module.

(iii) M is projective.

Hint. To prove (ii)⇒(i), you can assume (by Exercise 26) that R is local. Then lift a suitably chosen basis of $(M \oplus N) \otimes_R R/\mathfrak{m}$ to $M \oplus N$. To prove (i)⇒(iii), show that you can assume by Exercise 28 that R is local, and hence that M is free over R.

Exercise 30

Let X be a scheme (or any ringed space), and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type.

1. Let $x \in X$, let $U \subseteq X$ be an open neighborhood of x, and consider $s_i \in \Gamma(U, \mathscr{F})$, $i = 1, \dots, n$, such that the elements $(s_i)_x$ generate the stalk $\mathscr{F}_x$ as an $\mathscr{O}_X$-module. Prove that there exists an open neighborhood $V \subseteq U$ of x such that the $(s_i)_{|V}$ generate $\mathscr{F}_{|V}$.

2. Prove that the support

$$\operatorname{supp}(\mathscr{F}) := \{x \in X;\ \mathscr{F}_x \neq 0\}$$

is closed in X.