Prof. Dr. Ulrich Görtz **SS 2013**
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Algebraic Geometry 2, Exercises

Sheet 11, due June 28

Exercise 31

Let X be a scheme. For an $\mathscr{O}_X$-module $\mathscr{M}$ we denote by $\mathscr{M}^\vee := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)$ the dual $\mathscr{O}_X$-module.

1. Show that for every $\mathscr{O}_X$-module $\mathscr{M}$ there is a natural homomorphism $\mathscr{M} \to \mathscr{M}^{\vee\vee}$. Show that this homomorphism is an isomorphism if $\mathscr{M}$ is locally free of finite rank. Give an example of a scheme X and an $\mathscr{O}_X$-module of finite type, where this homomorphism is not an isomorphism.

2. Assume that $\mathscr{M}$ is an invertible $\mathscr{O}_X$-module. Show that there are isomorphisms
$$\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{M}^\vee \cong \mathscr{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{M}) \cong \mathscr{O}_X.$$
(*Hint:* Define homomorphisms of $\mathscr{O}_X$-modules between the objects above; then it can be checked locally on X that they are isomorphisms, i.e. one can assume that $\mathscr{M} \cong \mathscr{O}_X$.)

   Conclude that the set of isomorphism classes of invertible $\mathscr{O}_X$-modules is a group, where the multiplication is given by $\otimes_{\mathscr{O}_X}$, and the inverse is given by $\cdot^\vee$.

Exercise 32

Let R be principal ideal domain, $X = \operatorname{Spec} R$. Prove that every invertible $\mathscr{O}_X$-module is free.

Hint: Let M be a locally free R-module of rank 1. Prove that the natural homomorphism $M \to M \otimes_R K \cong K$ is injective, and conclude that M is isomorphic, as an R-module, to an ideal of R. (Similarly one can show that for X the spectrum of a Dedekind domain the group of isomorphism classes of invertible $\mathscr{O}_X$-modules is isomorphic to the class group, i.e. to the quotient of the group of all fractional ideals by the group of all principal ideals.)

Exercise 33

Let $f\colon X \to Y$ be a morphism of schemes. Let $\mathscr{F}$ be a sheaf on X, and let $\mathscr{G}$ be a sheaf on Y. Write down "the" natural maps between

$$\operatorname{Hom}(f^{-1}\mathscr{G}, \mathscr{F}) \qquad \text{and} \qquad \operatorname{Hom}(\mathscr{G}, f_*\mathscr{F})$$

and show that they are inverse to each other. (With some more work, one can show that the resulting bijections are functorial in $\mathscr{F}$ and $\mathscr{G}$, i.e., that f^{-1} and f_* are adjoint functors.)