Prof. Dr. Ulrich Görtz **SS 2013**
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Algebraic Geometry 2, Exercises

Sheet 12, due July 5

Exercise 34

Give an example of a scheme X, $n > 1$, and an $\mathscr{O}_X$-submodule $\mathscr{L} \subset \mathscr{O}_X^n$ such that $\mathscr{L}$ is invertible, but the quotient $\mathscr{O}_X^n/\mathscr{L}$ is not locally free.

Exercise 35

Let X be a scheme, $n > 1$, and let $\mathscr{L} \subset \mathscr{O}_X^n$ be an invertible $\mathscr{O}_X$-submodule such that the quotient $\mathscr{O}_X^n/\mathscr{L}$ is locally free. Denote by $\alpha\colon \mathscr{L} \to \mathscr{O}_X^n$ the inclusion, and by $\pi_i\colon \mathscr{O}_X^n \to \mathscr{O}_X$ the projection onto the i-th factor. For $i = 1, \dots, n$ define

$$U_i = \{x \in X;\ \pi_{i,x} \circ \alpha_x \text{ is an isomorphism } \mathscr{L}_x \cong \mathscr{O}_{X,x}\}$$

Prove that U_i is open in X and that $X = \bigcup_i U_i$.

Exercise 36

Let k be a field, $n \geq 1$. Recall that the group $PGL_{n+1}(k) := GL_{n+1}(k)/D$, where $D \cong k^\times$ is the subgroup of scalar diagonal matrices, acts by linear transformations on $\mathbb{P}_k^n$. Prove that $PGL_{n+1}(k)$ is equal to the group of automorphisms of $\mathbb{P}_k^n$ (as a k-scheme).

Hint. First prove that for every automorphism α of $\mathbb{P}_k^n$, we have $\alpha^*\mathscr{O}(1) \cong \mathscr{O}(1)$, using the fact that every line bundle on $\mathbb{P}_k^n$ is isomorphic to some $\mathscr{O}(i)$, $i \in \mathbb{Z}$.