Prof. Dr. Ulrich Görtz **SS 2013**
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Algebraic Geometry 2, Exercises

Sheet 13, due July 12

Exercise 37

Let X be a scheme and let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module.

1. Let $s \in \Gamma(X, \mathscr{L})$. Show that $X_s := \{x \in X;\ s(x) \in \mathscr{L}_x \setminus \mathfrak{m}_x \mathscr{L}_x\}$ is open in X. Show that $X_s \neq \emptyset$ whenever X is reduced and $s \neq 0$. Show that $\mathscr{L}_{|X_s}$ is a free $\mathscr{O}_{X_s}$-module.
2. Prove that every surjective homomorphism between locally free $\mathscr{O}_X$-modules of the same finite rank is an isomorphism. What about injective homomorphisms?
3. Let X be integral and assume that $\Gamma(X, \mathscr{O}_X)$ is a field. Assume that $\Gamma(X, \mathscr{L}) \neq 0$ and $\Gamma(X, \mathscr{L}^\vee) \neq 0$. Prove that $\mathscr{L} \cong \mathscr{O}_X$.

Exercise 38

Let k be a field. For a two-dimensional sub-vector space $U \subseteq k^4$ generated by the columns of the matrix $(a_{ij})_{i,j} \in M_{4\times 2}(k)$ we denote by $m_{i,i'}$ $(1 \leq i < i' \leq 4)$ the determinant of the matrix

$$\begin{pmatrix} a_{i1} & a_{i2} \\ a_{i'1} & a_{i'2} \end{pmatrix}$$

Prove that $(m_{12} : m_{13} : m_{14} : m_{23} : m_{24} : m_{34})$ is a point in $\mathbb{P}^5(k)$ which is independent of the choice of basis of U. Prove that in this way one obtains a closed immersion $\iota\colon \mathrm{Grass}_{2,4} \to \mathbb{P}^n_k$ of k-schemes which identifies $\mathrm{Grass}_{2,4}$ with $V_+(X_0X_5 - X_1X_4 + X_2X_3)$.

Exercise 39

We use the notation of Exercise 38. Let k be an algebraically closed field. We write $\mathcal{G} := \mathrm{Grass}_{2,4}$ and consider its closed points as lines in $\mathbb{P}^3_k$.

1. Let $p \in \mathbb{P}^3(k)$, and let $H \subset \mathbb{P}^3_k$ be a plane which contains p. Let $\Sigma_{p,H} \subset \mathcal{G}$ denote the set of lines in $\mathbb{P}^3_k$ which contain p and lie in H. Prove that under ι the set $\Sigma_{p,H}$ is identified with a line in $\mathbb{P}^5_k$, and that all lines in $\mathbb{P}^5_k$ which are contained in the image of ι are obtained in this way.

2. For $p \in \mathbb{P}^3(k)$ let Σ_p be the set of all lines in $\mathbb{P}^3_k$ containing p. For a plane $H \subset \mathbb{P}^3_k$, let Σ_H be the set of all lines in $\mathbb{P}^3_k$ lying in H. Prove that sets of the form Σ_p (or of the form Σ_H) are identified under ι with planes in $\mathbb{P}^5_k$, and that every plane contained in the image of ι is obtained as the image of a Σ_p or a Σ_H, for suitable p or H, respectively.