**Prof. Dr. Ulrich Görtz** **SS 2013**
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# Algebraic Geometry 2, Exercises

*Sheet 2, due 26.4.*

## Exercise 4

1. Let $k$ be a field, $A := k[X,Y]/(XY, Y^2)$. Define two $k$-algebra homomorphisms $\varphi_i\colon k[T]/(T^2) \to A$ for $i = 1,2$ by $\varphi_1(T) = 0$ and $\varphi_2(T) = Y$. Let $f_i\colon \operatorname{Spec} A \to \operatorname{Spec} k[T]/(T^2)$ be the associated morphisms of schemes. Show that there exists an open dense subset $U \subseteq \operatorname{Spec} A$ such that $f_{1|U} = f_{2|U}$ although $f_1 \neq f_2$. Determine $\operatorname{Ker}(f_1, f_2)$.
2. Give an example of a scheme $X$ and two affine open subschemes $U_1$, $U_2$ in $X$, such that $U_1 \cap U_2$ is not affine. (Can you find an example where $U_1 \cap U_2$ cannot be covered by finitely many affine open subschemes?)

## Exercise 5

Let $S$ be a scheme, let $X$, $Y$ be $S$-schemes. Assume that $X$ is integral, and denote by $\eta$ its generic point. Assume that $Y$ is separated over $S$. Let $f, g\colon X \to Y$ be morphisms such that $f(\eta) = g(\eta) := \vartheta$, and such that the homomorphisms $\kappa(\vartheta) \to \kappa(\eta)$ induced by $f$ and $g$ coincide. Show that $f = g$.

## Exercise 6

Let $S$ be a scheme. Let $X$, $Y$ be $S$-schemes. Assume that $X$ is integral, and that $Y$ is separated over $S$.

1. Denote by $\mathcal{R}(X,Y)$ the set of pairs $(U, f)$, where $\emptyset \neq U \subseteq X$ is open and $f\colon U \to Y$ is an $S$-morphism. Let
$$(U, f) \sim (V, g) \quad :\Leftrightarrow \quad f_{|U\cap V} = g_{|U\cap V}.$$
Prove that $\sim$ is an equivalence relation. The equivalence classes for this equivalence relation are called *rational $S$-maps* (or just rational maps, if $S$ is understood), notation: $X \dashrightarrow Y$.
2. Let $f\colon X \dashrightarrow Y$ be a rational $S$-map. Let $\operatorname{dom}(f)$ be the union of all sets $U$, where $(U, \dot{f})$ runs through all representatives of $f$. Prove that there exists a unique representative of $f$ of the form $(\operatorname{dom}(f), \dot{f})$.
3. Prove that there is a natural bijection between the rational function field $K(X)$ of $X$ and the set of rational $S$-maps $X \dashrightarrow \mathbb{A}^1_S$.