Prof. Dr. Ulrich Görtz **SS 2013**
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Algebraic Geometry 2, Exercises

Sheet 3, due May 3

Exercise 7

Let X be a topological space.

1. Let Y be a subspace of X. Then $\dim Y \leq \dim X$. If X is irreducible and finite-dimensional, and $Y \subsetneq X$ is closed, then $\dim Y < \dim X$.
2. Let $X = \bigcup_\alpha U_\alpha$ be an open covering. Then
$$\dim X = \sup_\alpha \dim U_\alpha.$$
3. Let I be the set of irreducible components of X. Then
$$\dim X = \sup_{Y \in I} \dim U$$
4. Now suppose that X is a scheme. Then
$$\dim X = \sup_{x \in X} \dim \mathscr{O}_{X,x}$$

Exercise 8

1. Let k be a field, let $X = \mathbb{A}_k^2$ with coordinates T, U, and let $Z = V(TU, T^2)$. Show that $\dim Z = 1$, and that there exists no $f \in k[T, U]$ with $Z = V(f)$.
2. Let R be a discrete valuation ring, and let $A = R[T]$. Prove that $\dim A = 2$. Give examples of maximal ideals $\mathfrak{m} \subset A$ such that $\dim A_\mathfrak{m} = 1$ and $\dim A_\mathfrak{m} = 2$, respectively.

Exercise 9

Let k be a field of characteristic $\neq 2$, let $n \geq 1$, and consider the following subschemes of $M_{n\times n,k} = \mathbb{A}_k^{n^2}$:

$$SO_n = \{M \in GL_{n,k};\ {}^tMM = I_n\},$$
$$\mathfrak{so}_n = \{M \in M_{n\times n,k};\ {}^tM = -M\}.$$

Here I denotes the $n \times n$ unit matrix.

1. Show that $\mathfrak{so}_n \cong \mathbb{A}_k^{n(n-1)/2}$.

2. Show that $A \mapsto (I_n + A)^{-1}(I_n - A)$ defines an isomorphism between a dense open subscheme of $\mathfrak{so}_n$ and a dense open subscheme of SO_n.

3. Deduce that SO_n is irreducible and that $\dim SO_n = \dim \mathfrak{so}_n$.