Prof. Dr. Ulrich Görtz **SS 2013**
Haifeng Wu

Algebraic Geometry 2, Exercises

Sheet 4, due May 10

Exercise 10

Let $f\colon X \to Y$ be a *closed* morphism of schemes (i.e. for all closed subsets $Z \subseteq X$, $f(Z)$ is closed in Y). Show that if f is surjective, then $\dim X \geq \dim Y$. Give an example which shows that the condition "f closed" cannot be omitted.

Exercise 11

1. Let $\varphi\colon A \to B$ be a ring homomorphism, and let $\mathfrak{p} \subset A$ be a prime ideal. Let $\mathfrak{P} \subseteq B$ be the ideal generated by $\varphi(\mathfrak{p})$. Assume that $\varphi^{-1}(\mathfrak{P}) = \mathfrak{p}$. Show that there exists a prime ideal $\mathfrak{q} \subset B$ with $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. (*Hint:* Let $S = \varphi(A \backslash \mathfrak{p})$. Show that there exists a maximal ideal $\mathfrak{m} \subset S^{-1}B$ containing $S^{-1}\mathfrak{P}$, and that its inverse image $\mathfrak{q}$ in B has the desired property.)
2. ("Going-down") Let $\varphi\colon A \to B$ be a finite injective ring homomorphism between integral domains. Let $\mathfrak{p}_1 \subset \mathfrak{p}_2$ be prime ideals of B, and let $\mathfrak{q}_2 \subset B$ be a prime ideal such that $\varphi^{-1}(\mathfrak{q}_2) = \mathfrak{p}_2$. Show that there exists a prime ideal $\mathfrak{q}_1 \subset B$ such that $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$ and $\varphi^{-1}(\mathfrak{q}_1) = \mathfrak{p}_1$.

   Hint: Show that $\varphi^{-1}(\mathfrak{p}_1 B_{\mathfrak{q}_2}) = \mathfrak{p}_1$ and apply part (1).

Exercise 12

Let k be a field, and let X be a non-empty k-scheme of finite type, and denote by $|X|$ its underlying topological space. Show that the following are equivalent:

(i) $|X|$ is discrete.

(ii) $|X|$ is finite.

(iii) $\dim X = 0$.

(iv) X is affine, $\dim_k \Gamma(X, \mathscr{O}_X)$ is finite, and

$$\Gamma(X, \mathscr{O}_X) = \prod_{x \in X} \mathscr{O}_{X,x}.$$

Hint: Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), that (i), (ii), (iii) together imply (iv), and that (iv) $\Rightarrow$ (i).