Prof. Dr. Ulrich Görtz **SS 2013**
Haifeng Wu

Algebraic Geometry 2, Exercises

Sheet 5, due May 17

Exercise 13

Let k be a field, and let X, Y be k-schemes locally of finite type. Prove that

$$\dim X \times_{\operatorname{Spec} k} Y = \dim X + \dim Y.$$

Exercise 14

Let k be a field.

1. Let $X \subseteq \mathbb{P}^n_k$ be an integral closed subscheme, and let $C(X) \subseteq \mathbb{A}^{n+1}_k$ be the cone over X, i.e., the closure in $\mathbb{A}^{n+1}_k$ of its inverse image under the projection $p\colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n_k$. Then $C(X) = p^{-1}(X) \cup \{0\}$. We denote by $\pi\colon C(X) \setminus \{0\} \to X$ the projection. Denote the homogeneous coordinates of $\mathbb{P}^n_k$ by $X_0, \dots, X_n$.

   Show that for all i we have

   $$\pi^{-1}(X \cap D_+(X_i)) \cong (X \cap D_+(X_i)) \times_{\operatorname{Spec} k} (\mathbb{A}^1_k \setminus \{0\}).$$

   Conclude that $\dim C(X) = \dim X + 1$.

2. Let $X \subseteq \mathbb{P}^n_k$ be an integral closed subscheme of dimension > 0, and let $f \in k[X_0, \dots, X_n]$ be a homogeneous polynomial such that $V_+(f) \neq \emptyset$ and $X \not\subseteq V_+(f)$. Show that then $X \cap V + (f) \neq \emptyset$ and that $X \cap V_+(f)$ is equi-codimensional of codimension 1 in X. (*Hint:* Consider $C(X) \cap V(f)$.)

Exercise 15

Let k be an algebraically closed field. Give examples of k-schemes X_1, X_2, X_3, X_4, such that for all four schemes there exists a morphism $\mathbb{A}^1_k \to X_i$ of k-schemes which is a homeomorphism, and such that

1. for all closed points $x \in X_1$, we have $\dim T_x(X_1) = 1$,
2. for all except exactly one closed point $x \in X_2$, we have $\dim T_x(X_2) = 1$, and X_2 is reduced,
3. for all except exactly one closed point $x \in X_3$, we have $\dim T_x(X_3) = 1$, and X_3 is not reduced,
4. for all closed points $x \in X_4$, we have $\dim T_x(X_4) > 1$.