Prof. Dr. Ulrich Görtz **SS 2013**
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Algebraic Geometry 2, Exercises

Sheet 6, due May 24

Exercise 16

Let k be a field, $n \geq 1$.

1. Let $x = (x_0\colon \cdots : x_n) \in \mathbb{P}^n(k)$. Show that the morphism $\mathbb{A}_k^{n-1} \setminus \{0\} \to \mathbb{P}_k^n$ induces a surjection $k^{n+1} = T_{(x_0,\dots,x_n)}(\mathbb{A}_k^{n+1} \setminus \{0\}) \to T_x\mathbb{P}_k^n$, and show that its kernel is the line generated by ${}^t(x_0, \dots, x_n)$.

2. Let $X = V_+(f_1, \dots, f_r) \subseteq \mathbb{P}_k^n$ be a closed subscheme (for homogeneous polynomials $f_i \in k[T_0, \dots, T_n]$), and let $x \in X(k)$. Prove that

$$T_xX \cong \left(\operatorname{Ker}\left(\frac{\partial f_i}{\partial X_j}(x)\right)_{i,j}\right)/kx \qquad (\subseteq T_x\mathbb{P}_k^n).$$

Exercise 17

Let k be a field, let X, Y be k-schemes of finite type, and let $x \in X(k)$, $y \in Y(k)$. Prove that

$$T_{(x,y)}(X \times_{\operatorname{Spec} k} Y) \cong T_xX \oplus T_yY.$$

Hint: Use the description of the tangent space in terms of $k[\varepsilon]/\varepsilon^2$-valued points.

Exercise 18

Let k be a field, and let $G = \mathrm{GL}_{n,k} = D(\det) \subset \mathbb{A}_k^{n^2}$. (We think of the points of $\mathbb{A}_k^{n^2}$ as $(n \times n)$-matrices.) Denote by $e \in \mathrm{GL}_n(k)$ the unit matrix. We identify $T_e\,\mathrm{GL}_{n,k} = k^{n^2}$ with the space $M_n(k)$ of $(n \times n)$-matrices.

(a) Denote by $m\colon G \times G \to G$ the multiplication map. Show that the homomorphism $df_{(e,e)}\colon T_eG \times T_eG = T_{(e,e)}(G \times G) \to T_eG$ is given by $(v, w) \mapsto v + w$.

(b) Show that the k-linear map $d\det_e\colon M_n(k) \to k$ induced by $\det\colon \mathrm{GL}_{n,k} \to \mathrm{GL}_{1,k}$ is the trace.