Prof. Dr. Ulrich Görtz **SS 2013**
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Algebraic Geometry 2, Exercises

Sheet 9, due June 14

Exercise 25

Let $\mathcal{C}$, $\mathcal{D}$ be categories, and let $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{F}$ be functors. We say that F is left adjoint to G (or that G is right adjoint to F), if there exist, for all objects X of $\mathcal{C}$ and Y of $\mathcal{D}$, bijections

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \qquad (*)$$

which are functorial in X and Y (i.e., for all morphisms $X' \to X$ in $\mathcal{C}$ and $Y' \to Y$ in $\mathcal{D}$ the obvious diagram commutes).

1. Assume that F is left adjoint to G. Show that (*) (applied with $Y = F(X)$, and $X = G(Y)$, respectively) induces morphisms $\varepsilon\colon \mathrm{id}_{\mathcal{C}} \to G \circ F$ and $\eta\colon F \circ G \to \mathrm{id}_{\mathcal{D}}$ of functors, such that the map (*) is given by $\alpha \mapsto G(\alpha) \circ \varepsilon(X)$, and its inverse is given by $\beta \mapsto \eta(Y) \circ F(\beta)$.

2. Let G and G' be functors which are both right adjoint to F. Prove that there is an isomorphism $G \cong G'$ of functors. (An analogous statement holds for left adjoint functors.)

3. Give the map (*) and its inverse in some of the following cases of adjoint functors (or find other cases of adjoint functors). Let R be a ring, and let $(R\mathrm{-mod})$ be the category of R-modules.

   (a) For any set I, we write $R^{(I)} = \bigoplus_{i \in I} R$ for the free R-module with standard basis indexed by I. Let $F\colon (\mathrm{Sets}) \to (R\mathrm{-mod})$, $I \mapsto R^{(I)}$, and let $G\colon (R\mathrm{-mod}) \to (\mathrm{Sets})$ be the forgetful functor.

   (b) Let $\mathcal{C}$ be the category of R-algebras, consider $F\colon (\mathrm{Sets}) \to \mathcal{C}$ given by $I \mapsto R[X_i;\ i \in I]$, and let $G\colon \mathcal{C} \to (\mathrm{Sets})$ be the forgetful functor.

   (c) Let P be an R-module, let $F\colon (R\mathrm{-mod}) \to (R\mathrm{-mod})$, $M \mapsto M \otimes_R P$, $G\colon (R\mathrm{-mod}) \to (R\mathrm{-mod})$, $N \mapsto \mathrm{Hom}_R(P, N)$.

   (d) Let R' be an R-algebra, consider $F\colon (R\mathrm{-mod}) \to (R'\mathrm{-mod})$, $M \mapsto M \otimes_R R'$, and let $G\colon (R'\mathrm{-mod}) \to (R\mathrm{-mod})$ which maps N to N, considered as an R-module.

   (e) Let X be a topological space, let $\mathcal{C}$ be the category of presheaves on X, let $\mathcal{D}$ be the category of sheaves on X. Let $F\colon \mathcal{C} \to \mathcal{D}$ be the sheafification functor and $G\colon \mathcal{D} \to \mathcal{C}$ be the forgetful functor.

Exercise 26

Let R be a ring. Recall that an R-module M is called locally free (of finite rank) if there exist $f_1, \dots, f_n \in R$ with $(f_1, \dots, f_n) = R$ and such that for all i, the R_{f_i}-module M_{f_i} is free (of finite rank). We say that M is locally free of rank r, is for all i, M_{f_i} is free of rank r.

1. Let R be noetherian and let M be a finitely generated R-module. Prove that M is locally free if and only if for all $\mathfrak{p} \in \operatorname{Spec} R$, the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is free.

2. Give an example of a ring R and an R-module M which is locally free, but not free. (*Hint:* One possibility is to take as R a Dedekind domain which is not a principal ideal domain, and as M an ideal of R which is not principal.)

Exercise 27

Let R be a discrete valuation ring with field of fractions K, $X = \operatorname{Spec} R$, $U = \operatorname{Spec} K$ (regarded as an open subscheme of X). Which of the following $\mathscr{O}_X$-modules (where in each case, the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ is the "obvious" map, and the $\mathscr{O}_X$-module structure is the "obvious" one) is of the form $\widetilde{M}$ for an R-module M?

1. $\mathscr{F}(X) = 0$, $\mathscr{F}(U) = K$,
2. $\mathscr{F}(X) = K$, $\mathscr{F}(U) = K$,
3. $\mathscr{F}(X) = k$, $\mathscr{F}(U) = 0$,
4. $\mathscr{F}(X) = R$, $\mathscr{F}(U) = K$,
5. $\mathscr{F}(X) = R$, $\mathscr{F}(U) = 0$.