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Sommersemester 2016

ALGEBRAIC NUMBER THEORY II

Problem Set 1

Due date: 26/4/2016

Exercise 1.

1. Let E/K be a Galois extension and let M/K be a Galois subextension of E/K. Let $\mathfrak{p}$ be a nonzero prime ideal of K, let $\mathfrak{P}$ be a prime ideal of E lying over $\mathfrak{p}$, and write $\mathfrak{p}_M = \mathfrak{P} \cap M$. Then $e_{M/K}(\mathfrak{p}_M) = f_{M/K}(\mathfrak{p}_M) = 1$ if and only if the decomposition group $D_{E/K}(\mathfrak{P}) \subseteq \mathrm{Gal}(E/M)$.

2. Let $\mathfrak{p}$ be a nonzero prime ideal of a number field K. Let L/K and L'/K be finite Galois extensions. Show that $\mathfrak{p}$ is split in LL'/K if and only if it is split in L/K and L'/K.

Exercise 2. Show that the polynomial $x^3 - 3x^2 + 2x + 3 \in \mathbb{Z}_3[x]$ decomposes into linear factors over $\mathbb{Q}_3$.

Exercise 3. Let $K/\mathbb{Q}_p$ be a finite extension, and denote by $\mathfrak{p}$ the maximal ideal of the valuation ring of K. Write $p\mathcal{O}_K = \mathfrak{p}^e$. Write $U^{(n)} = 1 + \mathfrak{p}^n$ for $n \geq 1$.

i) Show that for $1 + x \in U^{(1)}$, the following series converges

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

ii) Show that for $x \in \mathfrak{p}^n$ with $n > \frac{e}{p-1}$, the following series converges

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Hint: Let v_p be the p-adic valuation of $\mathbb{Q}_p$. Show that if $\nu = \sum_{i=0}^{r} a_i p^i$, with $0 \leq a_i < p$, then $v_p(\nu!) = \frac{1}{p-1}\sum_{i=0}^{r} a_i(p^i - 1)$.

iii) For $n > \frac{e}{p-1}$ show that log maps $U^{(n)}$ into $\mathfrak{p}^n$ and exp maps $\mathfrak{p}^n$ into $U^{(n)}$.

Hint: Show that if $v_\mathfrak{p}$ denotes the normalized valuation of K, then $v_\mathfrak{p}(\log(1+x)) = v_\mathfrak{p}(x)$ and $v_\mathfrak{p}(\exp(x) - 1) = v_\mathfrak{p}(x)$ for $v_\mathfrak{p}(x) > \frac{e}{p-1}$.

iv) Deduce that for $n > \frac{e}{p-1}$, $\log\colon U^{(n)} \to \mathfrak{p}^n$ and $\exp\colon \mathfrak{p}^n \to U^{(n)}$ are mutually inverse isomorphisms.

Hint: Simply invoke the following identities of formal power series:

$$\log((1+X)(1+Y)) = \log(1+X) + \log(1+Y), \quad \exp(X+Y) = \exp(X)\exp(Y),$$

$$\exp(\log(1+X)) = 1 + X, \qquad \log(\exp(X)) = X\,.$$

Exercise 4. Let $((X_i)_{i\in I}, (f_{ij})_{i\leq j\in I})$ be an inverse system of topological spaces X_i and continuous maps $f_{ij}\colon X_j \to X_i$. The inverse limit $X = \varprojlim X_i$ is endowed with projection maps

$$p_i\colon X \to X_i\,.$$

Equip X with the following topology: $U \subseteq X$ is an open set if and only if U is a union of subsets of the form $p_{i_1}^{-1}(U_{i_1}) \cap \cdots \cap p_{i_n}^{-1}(U_{i_n})$ for $i_\nu \in I$ and $U_{i_\nu} \subseteq X_{i_\nu}$ open.

i) Show that this is the coarsest topology such that all maps p_i are continuous.

ii) Show that if Y is a topological space and $g_i\colon Y \to X_i$ are continuous maps such that $g_i = f_{ij} \circ g_j$, then there exists a unique continuous map $u\colon Y \to X$ such that $g_i = p_i \circ u$.