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ALGEBRAIC NUMBER THEORY II

Problem Set 10

Due date: 28/6/2016

Exercise 1. Let G be a group and A a (not necessarily commutative) group on which G operates by group isomorphisms. We write A multiplicatively, and by abuse of notation call A a non-commutative G-module. A map $\varphi\colon G \mapsto A$ is called a 1-*cocycle* if $\varphi(\sigma\tau) = \varphi(\sigma) \cdot \sigma(\varphi(\tau))$ for every $\sigma, \tau \in G$. We denote by $Z^1(G, A)$ the set of all 1-cocycles. Say that two 1-cocycles φ, ψ are cohomologous, and write $\varphi \sim \psi$, if there exists $a \in A$ such that $\psi(\sigma) = a^{-1} \cdot \phi(\sigma) \cdot \sigma(a)$ for every $\sigma \in G$. Show that $\sim$ is an equivalence relation. The first cohomology group of A is the set of cohomology classes

$$H^1(G, A) = Z^1(G, A)/\sim\ .$$

Note that $H^1(G, A)$ has the structure of a "pointed set", that is, a set with a distinguished element corresponding to the trivial 1-cocycle satisfying $\varphi(\sigma) = 1$ for every $\sigma \in G$.

Remark: Note that if A is abelian, $H^1(G, A)$ coincides with the group defined in Ex. 2 of PS5.

Exercise 2. Let

$$1 \to A \overset{i}{\to} B \overset{\pi}{\to} C \to 1$$

be a sequence of non-commutative G-modules.

i) Given $c \in C^G$, define

$$\delta(c)\colon G \to A\,, \qquad \delta(c)(\sigma) = i^{-1}(b^{-1} \cdot \sigma(b))\,,$$

where $b \in B$ is such that $\pi(b) = c$. Show that $\delta(c) \in Z^1(G, A)$ and that its cohomology class is independent of the choice of b.

ii) Show that the sequence of pointed sets

$$1 \to A^G \overset{i_0}{\to} B^G \overset{\pi_0}{\to} C^G \overset{\delta}{\to} H^1(G, A) \overset{i_1}{\to} H^1(G, B) \overset{\pi_1}{\to} H^1(G, C)$$

is exact.

Remark: By the kernel of a morphism of pointed sets we mean the preimage of the distinguished element of the target set. One can see any group as a pointed set by considering the set underlying the group together with the neutral element. Above, i_0, i_1, (resp. π_0, π_1) are the maps induced by i (resp. π).

Exercise 3. Let L/K be a finite Galois extension of fields with Galois group $G = \mathrm{Gal}(L/K)$. Consider the natural action of G on $\mathrm{GL}_n(L)$.

i) Show that $H^1(G, \mathrm{GL}_n(L)) = 1$.

Hint: Imitate the procedure of PS 5, Ex. 3. To show that given a 1*-cocycle* $\varphi\colon G \to \mathrm{GL}_n(L)$*, there exists* $C \in \mathrm{GL}_n(L)$ *such that* $B = \sum_{\sigma \in G} \varphi(\sigma) \cdot \sigma(C)$ *is invertible, show first that a linear form* $L^n \to L$ *which vanishes on the image of the map*

$$b\colon L^n \to L^n\,, \qquad b(x) := \sum_{\sigma \in G} \varphi(\sigma) \cdot \sigma(x)$$

must be zero on all of L^n*. In other words, the image of* b *generates* L^n *over* L*. Then, if* $x_1, \ldots, x_n \in L^n$ *are such that the* $y_i = b(x_i)$ *generate* L^n*, take* C *to be the matrix of the linear map that sends the canonical basis* e_i *to* x_i*.*

ii) Deduce that $H^1(G, \mathrm{SL}_n(L)) = 1$.

Exercise 4. Let L/K be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$. For $n \geq 1$, let G act naturally on L^n and by

$$\sigma(\psi)(x) = \sigma(\psi(\sigma^{-1}(x))) \qquad \text{for all } \psi \in \mathrm{Aut}(L^n),\ x \in L^n$$

on $\mathrm{Aut}(L^n)$.

i) Show that giving $\phi \in Z^1(G, \mathrm{GL}_n(L))$ is equivalent to giving a family of K-vector space isomorphisms $\psi_\sigma\colon L^n \to L^n$ satisfying

$$\psi_\sigma \psi_\tau = \psi_{\sigma\tau} \qquad \text{for all } \sigma, \tau \in G$$

and

$$\psi_\sigma(\alpha x) = \sigma(\alpha)\psi_\sigma(x) \qquad \text{for all } \alpha \in L,\ x \in L^n.$$

Hint: To construct the family of ϕ_σ *from the* 1*-cocycle* ϕ*, set* $\psi_\sigma := \phi(\sigma) \circ \sigma(\cdot)$*.*

ii) Note that we can endow L^n with a new action of G by letting $\sigma \in G$ send $x \in L^n$ to $\psi_\sigma(x) \in L^n$. Write V to denote L^n with this new G-module structure. Show that

$$V^G = \{v \in V \mid \psi_\sigma(v) = v \text{ for all } \sigma \in G\}$$

satisfies $\dim_K(V^G) = n$ (and hence the natural map $V^G \otimes_K L \to V$ is an isomorphism).