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Sommersemester 2016

ALGEBRAIC NUMBER THEORY II

**Problem Set 11**

Due date: 5/7/2016

**Exercise 1.** *Algebraic independence of characters*
Let $K$ be an infinite field, $L/K$ be a finite Galois extension, and $\sigma_1 = \mathrm{id}, \sigma_2, \dots, \sigma_n$ the Galois automorphisms of $L/K$. Let $f \in K[X_1, \dots, X_n]$ be such that

$$f(\sigma_1(\alpha), \dots, \sigma_n(\alpha)) = 0$$

for all $\alpha \in L$. Prove that $f = 0$.

*Hint: Fix a basis of $L$ as a $K$-vector space, and do a suitable change of coordinates so that you can use the following fact (which you may use without proof): Let $E$ be an infinite field, and let $g \in E[X_1, \dots, X_r]$ with $g(x_1, \dots, x_r) = 0$ for all $x_i \in E$. Then $g = 0$.*

**Exercise 2.** Apply algebraic independence of characters to give another proof of PS 10, Ex. 3 i) in the case that $K$ is infinite.

*Hint: With the notation introduced in the hint of that exercise, let $C = cE_n$, $c \in L^\times$, be a scalar matrix and consider $\det(B)$.*

**Exercise 3.** *Normal basis theorem*
Prove that if $L/K$ is a finite Galois extension with $K$ an infinite field (and notation as in Exercise 1), there exists $\alpha \in L^\times$ such that $\sigma_i(\alpha)$ is a basis of $L$ as a $K$-vector space.

*Hint: Consider the matrix $A = (a_{i,j}) \in M_n(K[X_1, \dots, X_n])$, where $a_{i,j} = X_k$ if $\sigma_i \circ \sigma_j = \sigma_k$. Show that $\det(A) \neq 0$ and use algebraic independence of characters to prove the existence of $\alpha \in L^\times$ such that $\det(B) \neq 0$, where $B = (\sigma_i \circ \sigma_j(\alpha)) \in M_n(L)$.*

*Remark: The statement is also true for finite fields.*

**Exercise 4.**

i) Let $A$ be a discrete $\widehat{\mathbb{Z}}$-module. Show that $H^2(\widehat{\mathbb{Z}}, A) = 0$, if $A$ is torsion (i.e., for all $a \in A$, there exists $n \in \mathbb{Z} \setminus \{0\}$ with $na = 0$).

   *Hint: Writing $A$ as an inductive limit of finite $\widehat{Z}$-modules reduces to the case that $A$ is finite. Denote by $N_n$ the norm of $\widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}} (\cong \mathbb{Z}/n\mathbb{Z})$. Now apply the fact below to identify $\varinjlim H^2(\widehat{\mathbb{Z}}/n\widehat{\mathbb{Z}}, A^{n\widehat{\mathbb{Z}}}) = \varinjlim A^{\widehat{\mathbb{Z}}}/N_n A$, with transition maps $A^{\widehat{\mathbb{Z}}}/N_m A \to A^{\widehat{\mathbb{Z}}}/N_{mn} A$ given by multiplication by $n$.*

   **Fact.** Let $G$ be a cyclic group of order $n$, let $\sigma \in G$ be a generator, and let $\chi \in \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$ be defined by $\chi(\sigma) = 1/n$. Let $\theta = \delta(\chi) \in H^2(G, \mathbb{Z})$, where $\delta$ is the connecting homomorphism for the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Let $A$ be a $G$-module. Show that

$$H_T^0(G, A) \longrightarrow H_T^2(G, A), \qquad a \mapsto a \cup \theta,$$

is an isomorphism.

ii) Deduce that $H^2(\widehat{\mathbb{Z}}, A) = 0$ if $A$ is divisible (i.e., multiplication by $n$ is a surjection $A \to A$ for all $n \in \mathbb{Z} \setminus \{0\}$).

iii) Let $K$ be a perfect field with absolute Galois group $\mathrm{Gal}(\overline{K}/K) \cong \widehat{\mathbb{Z}}$. Show that the *Brauer group* $H^2(\mathrm{Gal}(\overline{K}/K), \overline{K}^\times)$ is trivial.

iv) Let $K$ be a perfect field with absolute Galois group $\mathrm{Gal}(\overline{K}/K) \cong \widehat{\mathbb{Z}}$. Show that for each finite extension $L/K$, the norm map $N_{L/K}$ is surjective.