Prof. Dr. U. Görtz
Dr. F. Fité

Sommersemester 2016

ALGEBRAIC NUMBER THEORY II

Problem Set 12

Due date: 12/7/2016

Exercise 1. Let $\zeta_1, \zeta_2 \in \overline{\mathbb{Q}}_p$ be roots of unity such that $\mathbb{Q}_p(\zeta_1)/\mathbb{Q}_p$ is unramified of degree f and ζ_2 has order p^m. Show that

$$N_{\mathbb{Q}_p(\zeta_1\zeta_2)/\mathbb{Q}_p}(\mathbb{Q}_p(\zeta_1\zeta_2)^\times) = \langle p^f \rangle \times U^{(m)} .$$

Exercise 2. *Local Kronecker-Weber*
Show that every finite abelian extension $K/\mathbb{Q}_p$ is contained in a field of the form $\mathbb{Q}_p(\zeta)$, where $\zeta \in \overline{\mathbb{Q}}_p$ is a root of unity.

Exercise 3. *Artin-Schreier theory*
Let k be a field of characteristic $p > 0$, $\overline{k}$ a separable closure of k, and $G = \mathrm{Gal}(\overline{k}/k)$. Prove that there is an exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to k \xrightarrow{\wp} k \to H^1(G, \mathbb{Z}/p\mathbb{Z}) \to 0 ,$$

where $\wp(x) = x^p - x$ and G acts trivially on $\mathbb{Z}/p\mathbb{Z}$.

Exercise 4. Let $K/\mathbb{Q}_p$ be a finite extension containing all mth roots of unity ($m \geq 1$). For $a \in K^\times$, set $L_a = K(\sqrt[m]{a})$, and let $L = K(\sqrt[m]{a};\ a \in K)$ be the compositum of all L_a. Prove that

$$(K^\times)^m = \bigcap_{a \in K^\times} N_{L_a/K}(L_a^\times) = N_{L/K} L^\times .$$

Hint: Use Kummer theory (PS 7, Ex. 4).