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Sommersemester 2016

ALGEBRAIC NUMBER THEORY II

Problem Set 2

Due date: 3/5/2016

Exercise 1. A *topological group* G is a topological space endowed with a group structure such that the operations of product

$$G \times G \to G\,, \qquad (x,y) \mapsto xy$$

and taking inverses

$$G \to G\,, \qquad x \mapsto x^{-1}$$

are continuous maps. Let G and G' be topological groups.

i) Show that a subgroup $H \subseteq G$ is open if and only if it contains an open neighbourhood of the identity element $1 \in G$.

ii) Show that a group homomorphism $f\colon G \to G'$ is continuous if and only if there is a basis of open neighbourhoods $\mathcal{B}$ of the identity $1 \in G'$ such that $f^{-1}(B)$ is open for every $B \in \mathcal{B}$.

iii) Show that every open subgroup of G is also closed.

iv) Give an example of a closed and non-open subgroup in a topological group.

Exercise 2. Let K be a field with a non-trivial non-archimedean absolute value $|\cdot|$. Suppose that K is locally compact with the topology induced by $|\cdot|$. Prove that then K is complete, $|\cdot|$ is discrete, and the residue field is finite.

Exercise 3. Let K be a local field. Prove that:

i) If $\mathrm{Char}(K) = 0$, then $(K^\times)^n$ is an open subgroup of $K^\times$ for every $n \geq 1$.

ii) If $\mathrm{Char}(K) = p$, then $(K^\times)^n$ is an open subgroup of $K^\times$ if and only if $p \nmid n$.

Hint: Use the "p-adic Newton method" in the form that given $f \in \mathcal{O}_K[X]$ and $a \in \mathcal{O}_K$ with $|f(a)| < |f'(a)|^2$ there exists a zero b of f in $\mathcal{O}_K$.

Exercise 4. Let K be a complete non-archimedean field and let $\overline{K}$ denote an algebraic closure. Let $\alpha, \beta \in \overline{K}$, assume that α is separable over $K(\beta)$, and let $\alpha = \alpha_1, \alpha_2, \dots, \alpha_n$ be the Galois conjugates of α over K. Prove that if

$$|\alpha - \beta| < |\alpha - \alpha_i|$$

for $2 \leq i \leq n$, then $K(\alpha) \subseteq K(\beta)$.

Hint: Note that it is enough to show that $\tau(\alpha) = \alpha$ for all $\tau \in \mathrm{Hom}_{K(\beta)}(K(\alpha,\beta), \overline{K})$.