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Sommersemester 2016

ALGEBRAIC NUMBER THEORY II

Problem Set 3

Due date: 10/5/2016

Exercise 1. Give an example of a field K, complete with respect to a non-archimedean absolute value and with perfect residue class field, and two totally ramified extensions L_1/K and L_2/K such that their compositum L_1L_2/K is not totally ramified.

Exercise 2.

i) Prove that there exists a sequence $(a_n)_{n\in\mathbb{N}}$, $a_n \in \mathbb{Z}$, satisfying

$$a_n \equiv a_m \pmod{m}$$

whenever $m|n$, but such that there is no $a \in \mathbb{Z}$ such that $a_n \equiv a \pmod{n}$ for every $n \in \mathbb{N}$.

ii) Now let $\mathbb{F}_q$ be a finite field, q some prime power. Deduce from part i) that $\mathrm{Frob}_q^{\mathbb{Z}} \subsetneq \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

Exercise 3. Let G be a profinite group and let G' be the closure of its commutator group $[G, G]$. Show that $G^{\mathrm{ab}} = G/G'$ is a profinite group and that every continuous homomorphism $G \to A$, where A is an abelian profinite group, factors through G^{ab}.

Exercise 4.

i) Let K be a local field of characteristic 0. Show that every subgroup of $K^\times$ of finite index is open.

ii) Let K be the extension field of $\mathbb{Q}$ generated by all $\sqrt{p}$ where p is a prime number. Show that $G := \mathrm{Gal}(K/\mathbb{Q}) \cong \prod_{i\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. Via this identification, let $H = \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} \subset G$. Show that H is dense in G. Prove that there exists a subgroup $H \subset H' \subset G$ such that G/H' is finite but non-trivial (choose a basis of the $\mathbb{F}_2$-vector space G/H in order to find such a H'). Conclude that G has normal subgroups of finite index which are not open. (*Remark.* It is easy to deduce that also $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ has normal subgroups of finite index which are not open.)