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Sommersemester 2016

ALGEBRAIC NUMBER THEORY II

Problem Set 4

Due date: 17/5/2016

Exercise 1. Let E/K be a finite Galois extension of local fields. Prove that $\mathrm{Gal}(E/K)$ is solvable.

Exercise 2. Let L/K be an unramified extension of non-archimedean local fields. Let $f = [L : K]$, let $\pi \in K$ be a uniformizer, and let κ and λ be the residue fields of K and L.

i) Let $n = |\lambda| - 1$ and $m = |\kappa| - 1$. The mth (resp. nth) roots of unity are contained in K (resp. L). Prove that for every mth root of unity ζ in K there is an nth root of unity ξ in L such that $N_{L/K}(\xi) = \zeta$.

   Hint: It is enough to show that if ξ is a primitive nth root of unity, then

$$N_{L/K}(\xi) = \xi^{1+|\kappa|+|\kappa|^2 \cdots + |\kappa|^{f-1}}$$

   is a primitive mth root of unity.

ii) Prove that $N_{L/K}(L^\times) = \langle \pi^f \rangle \times \mathcal{O}^\times$, where $\mathcal{O}$ denotes the ring of integers of K.

   Hint: The difficulty is showing that $\mathcal{O}^\times$ is contained in $N_{L/K}(L^\times)$. Let $\overline{\gamma}$ be a generator of $\lambda^\times$ and let $\overline{\alpha} = N_{\lambda/\kappa}(\overline{\gamma})$ be a generator of $\kappa^\times$. By i), it is enough to show that for every $\alpha \in \mathcal{O}^\times$ such that $\alpha \equiv \overline{\alpha} \pmod{\pi}$ there exists $\gamma \in L^\times$ such that $N_{L/K}(\gamma) = \alpha$. Let $f(T) \in \kappa[T]$ be the minimal polynomial of $\overline{\gamma}$ over κ. Show that you can obtain such a γ as a root of a lift of $f(T)$ in $K[T]$, whose constant term is $(-1)^f \alpha$.

Exercise 3. Let K be a non-archimedean local field, and denote by π a uniformizer of K. Let K^{un} and K^{tr} be its maximal unramified and maximal tamely ramified extensions. Denote by $I^{\mathrm{tr}} = \mathrm{Gal}(K^{\mathrm{tr}}/K^{\mathrm{ur}})$ the tame inertia group. Recall from the course that there is a canonical isomorphism

$$t_0 \colon I^{\mathrm{tr}} \xrightarrow{\cong} \varprojlim_{p \nmid e} \mu_e(K^{\mathrm{un}})$$

induced by the isomorphisms $\mathrm{Gal}(K(\sqrt[e]{\pi})/K) \cong \mu_e(K^{\mathrm{un}})$, $\sigma \mapsto \sigma(\sqrt[e]{\pi})/\sqrt[e]{\pi}$ (where $\sqrt[e]{\pi}$ denotes a fixed zero of $X^e - \pi$).

1. Let $\varphi \in \mathrm{Gal}(K^{\mathrm{un}}/K)$ be the Frobenius automorphism. Then φ acts on I^{tr} by conjugation (i.e., every lift of φ to an element of $\mathrm{Gal}(K^{\mathrm{tr}}/K)$ acts on the normal subgroup I^{tr} by conjugation, and the action is independent of the lift). Show that

$$t_0(\varphi\sigma\varphi^{-1}) = t_0(\sigma)^q.$$

2. Conclude that the extension K^{tr}/K is not abelian.

Exercise 4.

i) Give an example of a group G and an exact sequence of G-modules

$$0 \to A \to B \to C \to 0$$

such that

$$0 \to A^G \to B^G \to C^G \to 0$$

is not exact.

ii) Give an example of a group G and G-modules A and B such that the map

$$A^G \otimes B^G \to (A \otimes B)^G$$

is neither injective nor surjective.